Construct the Hamiltonian

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The notation used in this notebook corresponds to the notation introduced in the LaTeX notes "endpoint.tex" and to Simon's paper, hep−ph/0101318.

■ Definitions

```
Off[General::"spell1"]
```

All symbols are considered to be c−numberunless specified otherwise:

```
cnumberq[x_] := FreeQ[x, sigma[_]] && FreeQ[x, a[__]] && FreeQ[x, b[__]] && FreeQ[x, d[__]] && FreeQ[x, zeta[_]]
freelistq[x_, y_List] := Apply[And, Map[FreeQ[x, #] &, y]]

prodexpand = {(a__) ** b_Plus :> Map[(a ** #) &, b], b_Plus ** (a__) :> Map[(# ** a) &, b],
   a_ ** b_Times :> Select[b, cnumberq] a ** Select[b, ! cnumberq[#] &],
   b_Times ** a_ :> Select[b, cnumberq] Select[b, ! cnumberq[#] &] ** a};

intsumexpand = {a_ ** int[b_, c_] :> int[a ** b, c] /; freelistq[a, c], int[b_, c_] ** a_ :> int[b ** a, c] /; freelistq[a, c],
   a_ ** sum[b_, c_] :> sum[a ** b, c] /; freelistq[a, c], sum[b_, c_] ** a_ :> sum[b ** a, c] /; freelistq[a, c],
   int[a_Times, b_] :>
    Module[{vars = Select[a, freelistq[#, b] &]}, vars int[Select[a, ! freelistq[#, b] &], b] /; vars =!= 1],
   sum[a_Times, b_] :> Module[{vars = Select[a, freelistq[#, b] &]},
     vars sum[Select[a, ! freelistq[#, b] &], b] /; vars =!= 1],
   int[a_Plus, b_] :> Map[int[#, b] &, a],
   sum[a_Plus, b_] :> Map[sum[#, b] &, a],
   int[d_. sum[a_, b_], c_] -> sum[int[d a, c], b],
   int[a_. int[b_, c_], d_] :> int[Expand[a b], Join[c, d]] /; freelistq[a, c],
   sum[a_. sum[b_, c_], d_] :> sum[Expand[a b], Join[c, d]] /; freelistq[a, c],
   int[a_, b_] :> int[Expand[a], b] /; Expand[a] =!= a};

prodfactor = {a_. c_NonCommutativeMultiply + b_. c_NonCommutativeMultiply :> Simplify[Factor[Expand[a] + Expand[b]]] c};

sigmazeta = {a[b__] ** sigma[c_] → sigma[c] ** a[b], Conjugate[a[b__]] ** sigma[c_] → sigma[c] ** Conjugate[a[b]],
   a[b__] ** zeta[c_] → zeta[c] ** a[b], Conjugate[a[b__]] ** zeta[c_] → zeta[c] ** Conjugate[a[b]],
   a[b__] ** Conjugate[zeta[c_]] → Conjugate[zeta[c]] ** a[b],
   Conjugate[a[b__]] ** Conjugate[zeta[c_]] → Conjugate[zeta[c]] ** Conjugate[a[b]]};
```

Relabel the parameters for implicit sums and integrations. It acts on an expression that is linear in NonCommutativeMultiply[...].

```
relabel[a_Plus] := Map[relabel, a];
relabel[ain_bin_NonCommutativeMultiply] := Block[{i1 = 1, i2 = 1, j = 1, krules, kcrules, srules, ops, cops},
  ops = Cases[Apply[List, bin], a[_Symbol, _] | b[_Symbol, ___] | d[_Symbol, ___]];
  cops = Cases[Apply[List, bin], Conjugate[a[_Symbol, _] | b[_Symbol, ___] | d[_Symbol, ___]]];
  krules = Map[(First[#] → Symbol["kr" <> ToString[i2++]]) &, Reverse[ops]];
  kcrules = Map[(First[First[#]] → Symbol["kl" <> ToString[i1++]]) &, cops];
  srules = Map[(# → Symbol["s" <> ToString[j++]]) &, Union[Map[Last, Cases[
       Apply[List, bin] /. Conjugate[ex_] → ex /. a[_, -b_Symbol] → a[0, b], a[_, _Symbol] | zeta[_Symbol] | sigma[_Symbol]
      ]]]];
  ain bin /. Join[krules, kcrules] /. srules]
```

simplify an integration over x, assuming the variable is in a delta−function. It acts on an expression that is linear in delta[...] and NonCommutativeMultiply[...].

```
simpint[a_Plus, x_] := Map[simpint[#, x] &, a];
simpint[a_ b_, x_] := a simpint[b, x] /; FreeQ[a, x];
simpint[a_ delta[-x_ + c_], x_] := simpint[a delta[x - c], x];
simpint[a_ delta[x_ + c_], x_] := Simplify[a /. x → -c]
simpint[a_ , x_] := a /; FreeQ[a, x];

Unprotect[Conjugate];
Conjugate[a_Plus] := Map[Conjugate, a];
Conjugate[a_Times] := Map[Conjugate, a];
Conjugate[a_ ** b_] := Conjugate[b] ** Conjugate[a];
Protect[Conjugate];
sigma /: Conjugate[sigma[a_]] = sigma[a];
sigma /: sigma[a_Integer] ** sigma[a_Integer] ** b_ = b;
sigma /: b ** sigma[a_Integer] ** sigma[a_Integer] = b;
zeta /: Conjugate[zeta[a_]] := zeta[a];
int /: Conjugate[int[a_, b_]] := int[Conjugate[a] /. Map[(Conjugate[#] → #) &, b], b];
sum /: Conjugate[sum[a_, b_]] := sum[Conjugate[a] /. Map[(Conjugate[#] → #) &, b], b];

intconv[x_] := (int[a_, b_] :→ int[Integrate[a, x], Complement[b, {x}]] /; MemberQ[b, x])
```

A variable x is defined to be positive[x]=True if it is *always* positive.

```
Unprotect[Positive];
Positive[a_ + b_] := False /; ! Positive[a] && ! Positive[b];
Positive[a_?Positive + b_?Positive] = True;
Positive[a_?Negative b_?Positive] = False;
Protect[Positive];
neverzero[a_] := If[Positive[a], True, True, False];

On[General::"spell1"]
```

■ Find Yukawa terms (OLD)

■ Find terms in the Hamiltonian

This version leaves all of the integrations over momenta and position implicit. Also, sums are implicit.

```
u[x_] := 1 / Sqrt[2 Pi] (b[kf] Exp[-I kf x] + sigma[1] ** Conjugate[d[kf]] Exp[I kf x])
m[x_, rho_] := 1 / Sqrt[4 Pi] (a[ka, -rho] Exp[-I ka x] + Conjugate[a[ka, rho]] Exp[I ka x]) / Sqrt[ka]
kappaa /: Conjugate[kappaa] := kappaa
kappas /: Conjugate[kappas] := kappas
muf /: Conjugate[muf] := muf
```

This is to define the momenta variables to be *always * positive.

```
kl1 /: Positive[kl1] = True; kl2 /: Positive[kl2] = True; kl3 /: Positive[kl3] = True; kl4 /: Positive[kl4] = True;
kr1 /: Positive[kr1] = True; kr2 /: Positive[kr2] = True; kr3 /: Positive[kr3] = True; kr4 /: Positive[kr4] = True;
```

■ Find Yukawa terms

```
xi = Expand[muf / Sqrt[2] sigma[1] ** u[x] +
       kappaa / Sqrt[2] sigma[1] ** zeta[rr] ** m[x, rr] ** u[x] - kappas / Sqrt[2] sigma[1] ** m[x, sumrr] ** u[x] //. prodexpand /.
     {zeta[rho_] ** a[c_, -rho_] → -zeta[rho] ** a[c, rho], a[b_, -sumrr] → a[b, sumrr]} //. prodexpand]
```

$$\frac{e^{i\,kf\,x}\,muf\,\text{Conjugate}[d[kf]]}{2\sqrt{\pi}} + \frac{e^{-i\,kf\,x}\,muf\,sigma[1] ** b[kf]}{2\sqrt{\pi}} -$$

$$\frac{e^{-i\,ka\,x-i\,kf\,x}\,kappas\,sigma[1] ** a[ka, sumrr] ** b[kf]}{4\sqrt{ka}\,\pi} - \frac{e^{i\,ka\,x-i\,kf\,x}\,kappas\,sigma[1] ** \text{Conjugate}[a[ka, sumrr]] ** b[kf]}{4\sqrt{ka}\,\pi} -$$

$$\frac{e^{-i\,ka\,x+i\,kf\,x}\,kappas\,sigma[1] ** a[ka, sumrr] ** sigma[1] ** \text{Conjugate}[d[kf]]}{4\sqrt{ka}\,\pi} -$$

$$\frac{e^{i\,ka\,x+i\,kf\,x}\,kappas\,sigma[1] ** \text{Conjugate}[a[ka, sumrr]] ** sigma[1] ** \text{Conjugate}[d[kf]]}{4\sqrt{ka}\,\pi} -$$

$$\frac{e^{-i\,ka\,x-i\,kf\,x}\,kappaa\,sigma[1] ** zeta[rr] ** a[ka, rr] ** b[kf]}{4\sqrt{ka}\,\pi} +$$

$$\frac{e^{i\,ka\,x-i\,kf\,x}\,kappaa\,sigma[1] ** zeta[rr] ** \text{Conjugate}[a[ka, rr]] ** b[kf]}{4\sqrt{ka}\,\pi} -$$

$$\frac{e^{-i\,ka\,x+i\,kf\,x}\,kappaa\,sigma[1] ** zeta[rr] ** a[ka, rr] ** sigma[1] ** \text{Conjugate}[d[kf]]}{4\sqrt{ka}\,\pi} +$$

$$\frac{e^{i\,ka\,x+i\,kf\,x}\,kappaa\,sigma[1] ** zeta[rr] ** \text{Conjugate}[a[ka, rr]] ** sigma[1] ** \text{Conjugate}[d[kf]]}{4\sqrt{ka}\,\pi}$$

```
cxi = Conjugate[xi] /. {Conjugate[x] → x, Conjugate[ka] → ka, Conjugate[kf] → kf, Conjugate[ka^n_] → ka^n} /.
    {ka → kka, kf → kks, rr → ll, sumrr → sumll} //. prodexpand //. sigmazeta
```

$$\frac{e^{-i\,kks\,x}\,muf\,d[kks]}{2\sqrt{\pi}} + \frac{e^{i\,kks\,x}\,muf\,Conjugate[b[kks]]\;**\;sigma[1]}{2\sqrt{\pi}} - \frac{e^{-i\,kka\,x-i\,kks\,x}\,kappas\,d[kks]\;**\;a[kka,\,sumll]}{4\sqrt{kka}\,\pi} -$$
$$\frac{e^{i\,kka\,x-i\,kks\,x}\,kappas\,d[kks]\;**\;Conjugate[a[kka,\,sumll]]}{4\sqrt{kka}\,\pi} - \frac{e^{-i\,kka\,x+i\,kks\,x}\,kappas\,Conjugate[b[kks]]\;**\;sigma[1]\;**\;a[kka,\,sumll]}{4\sqrt{kka}\,\pi} -$$
$$\frac{e^{i\,kka\,x+i\,kks\,x}\,kappas\,Conjugate[b[kks]]\;**\;sigma[1]\;**\;Conjugate[a[kka,\,sumll]]}{4\sqrt{kka}\,\pi} +$$
$$\frac{e^{-i\,kka\,x+i\,kks\,x}\,kappaa\,Conjugate[b[kks]]\;**\;zeta[ll]\;**\;sigma[1]\;**\;a[kka,\,ll]}{4\sqrt{kka}\,\pi} -$$
$$\frac{e^{i\,kka\,x+i\,kks\,x}\,kappaa\,Conjugate[b[kks]]\;**\;zeta[ll]\;**\;sigma[1]\;**\;Conjugate[a[kka,\,ll]]}{4\sqrt{kka}\,\pi} +$$
$$\frac{e^{-i\,kka\,x-i\,kks\,x}\,kappaa\,d[kks]\;**\;sigma[1]\;**\;zeta[ll]\;**\;sigma[1]\;**\;a[kka,\,ll]}{4\sqrt{kka}\,\pi} -$$
$$\frac{e^{i\,kka\,x-i\,kks\,x}\,kappaa\,d[kks]\;**\;sigma[1]\;**\;zeta[ll]\;**\;sigma[1]\;**\;Conjugate[a[kka,\,ll]]}{4\sqrt{kka}\,\pi}$$

```
expr1 = Collect[relabel[Expand[(-I cxi ** Integrate[xi, x] + I Integrate[cxi, x] ** xi) / 2 //. prodexpand //. sigmazeta]],
    {muf, kappaa, kappas}] //. prodfactor;
```

Since the delta[...] is removed in the following, the integration region can be complicated. expr2 contians all of the interactions in the Hamiltonian.

```
expr2 = simpint[int[expr1, {x}] //. intsumexpand /. Exp[x_] :→ Exp[Factor[x]] /.
    int[Exp[a_ b_], {b_}] → 2 Pi delta[I a] /. delta[_?neverzero] → 0, kl1]
```

These are the 1→1 particle interactions:

```
expr2 /. a_NonCommutativeMultiply :→
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 2 || Count[non, Conjugate[_]] != 1]
```

$$muf^2\left(\frac{(kl1+kr1)\,delta[-kl1+kr1]\,Conjugate[b[kl1]]\;**\;b[kr1]}{4\,kl1\,kr1} - \frac{(kl1+kr1)\,delta[-kl1+kr1]\,d[kr1]\;**\;Conjugate[d[kl1]]}{4\,kl1\,kr1}\right)$$

These are the 2→1 particle interactions

```
expr2 /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 3 || Count[non, Conjugate[_]] != 1]
```

```
muf
 (kappas (((kr1 kr2 (kr1 + kr2) - kl1^2 (2 kr1 + kr2) + kl1 (-2 kr1^2 + kr2^2)) delta[-kl1 + kr1 + kr2] Conjugate[b[kl1]] ** a[kr2, s1] **
          b[kr1]) / (8 kl1 kr1 (kl1 - kr2) Sqrt[kr2] (kr1 + kr2) Sqrt[π]) +
      ((kl1 + kr2) (-kr1 (kr1 + kr2) + kl1 (kr1 + 2 kr2)) delta[-kl1 + kr1 + kr2] d[kr2] ** a[kr1, s1] ** Conjugate[d[kl1]]) /
       (8 kl1 (kl1 - kr1) Sqrt[kr1] kr2 (kr1 + kr2) Sqrt[π]) + ((kr1 - kr2) (kl1^2 + 2 kr1 kr2 - kl1 (kr1 + kr2)) delta[-kl1 + kr1 + kr2]
         d[kr2] ** sigma[1] ** Conjugate[a[kl1, s1]] ** b[kr1]) / (8 Sqrt[kl1] (kl1 - kr1) kr1 (kl1 - kr2) kr2 Sqrt[π])) +
   kappaa ((Sqrt[kr2] (kl1^2 - kl1 kr2 + kr1 (kr1 + kr2)) delta[-kl1 + kr1 + kr2] Conjugate[b[kl1]] ** zeta[s1] ** a[kr2, s1] ** b[kr1]) /
       (8 kl1 kr1 (kl1 - kr2) (kr1 + kr2) Sqrt[π]) + (Sqrt[kl1] (kr1^2 + kr2^2 - kl1 (kr1 + kr2)) delta[-kl1 + kr1 + kr2]
         d[kr2] ** sigma[1] ** zeta[s1] ** Conjugate[a[kl1, s1]] ** b[kr1]) / (8 kr1 (-kl1 + kr1) kr2 (-kl1 + kr2) Sqrt[π]) +
      (Sqrt[kr1] (kl1^2 - kl1 kr1 + kr2 (kr1 + kr2)) delta[-kl1 + kr1 + kr2] d[kr2] ** sigma[1] ** zeta[s1] **
          sigma[1] ** a[kr1, s1] ** Conjugate[d[kl1]]) / (8 kl1 (kl1 - kr1) kr2 (kr1 + kr2) Sqrt[π])))
```

These are the 1→2 particle interactions:

```
expr2 /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 3 || Count[non, Conjugate[_]] ≠ 2]
```

```
muf
 (kappas (-((kl1^2 (kl2 - 2 kr1) + kl2 (kl2 - kr1) kr1 + kl1 (kl2^2 - 2 kr1^2)) delta[-kl1 - kl2 + kr1] Conjugate[b[kl1]] ** Conjugate[
           a[kl2, s1]] ** b[kr1]) / (8 kl1 Sqrt[kl2] (kl1 + kl2) (kl2 - kr1) kr1 Sqrt[π]) +
      ((kl2 + kr1) (kl1^2 + kl1 (kl2 - kr1) - 2 kl2 kr1) delta[-kl1 - kl2 + kr1] d[kr1] ** Conjugate[a[kl1, s1]] **
         Conjugate[d[kl2]]) / (8 Sqrt[kl1] kl2 (kl1 + kl2) (kl1 - kr1) kr1 Sqrt[π]) +
      ((kl1^2 (2 kl2 - kr1) + kl2 (kl2 - kr1) kr1 + kl1 (-2 kl2^2 + kr1^2)) delta[-kl1 - kl2 + kr1]
        Conjugate[b[kl1]] ** sigma[1] ** a[kr1, s1] ** Conjugate[d[kl2]]) / (8 kl1 kl2 (kl1 - kr1) (kl2 - kr1) Sqrt[kr1] Sqrt[π])) +
   kappaa (-(Sqrt[kl2] (kl1^2 + kl1 kl2 + kr1 (-kl2 + kr1)) delta[-kl1 - kl2 + kr1] Conjugate[b[kl1]] ** zeta[s1] **
          Conjugate[a[kl2, s1]] ** b[kr1]) / (8 kl1 (kl1 + kl2) (kl2 - kr1) kr1 Sqrt[π]) +
      (Sqrt[kr1] (kl1^2 + kl2 (kl2 - kr1) - kl1 kr1) delta[-kl1 - kl2 + kr1] Conjugate[b[kl1]] ** zeta[s1] **
         sigma[1] ** a[kr1, s1] ** Conjugate[d[kl2]]) / (8 kl1 kl2 (kl1 - kr1) (kl2 - kr1) Sqrt[π]) -
      (Sqrt[kl1] (kl2^2 + kl1 (kl2 - kr1) + kr1^2) delta[-kl1 - kl2 + kr1] d[kr1] ** sigma[1] ** zeta[s1] ** sigma[1] **
         Conjugate[a[kl1, s1]] ** Conjugate[d[kl2]]) / (8 kl2 (kl1 + kl2) (kl1 - kr1) kr1 Sqrt[π])))
```

These are the 3→1 particle interactions:

```
expr2 /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 1]
```

$$
\begin{aligned}
&\text{kappas}^2 \left(- \frac{(\text{kl1} + \text{kr1} + \text{kr2} - \text{kr3})\ \text{delta}[-\text{kl1} + \text{kr1} + \text{kr2} + \text{kr3}]\ \text{Conjugate}[\text{b}[\text{kl1}]] ** \text{a}[\text{kr3, s1}] ** \text{a}[\text{kr2, s2}] ** \text{b}[\text{kr1}]}{16\sqrt{\text{kr2}}\ (\text{kr1} + \text{kr2})\ \sqrt{\text{kr3}}\ (-\text{kl1} + \text{kr3})\ \pi} + \right. \\
&\quad \frac{(\text{kl1} - \text{kr1} + \text{kr2} + \text{kr3})\ \text{delta}[-\text{kl1} + \text{kr1} + \text{kr2} + \text{kr3}]\ \text{d}[\text{kr3}] ** \text{a}[\text{kr2, s1}] ** \text{a}[\text{kr1, s2}] ** \text{Conjugate}[\text{d}[\text{kl1}]]}{16\sqrt{\text{kr1}}\ (-\text{kl1} + \text{kr1})\ \sqrt{\text{kr2}}\ (\text{kr2} + \text{kr3})\ \pi} - \\
&\quad ((\text{kl1} - \text{kr1} + \text{kr2} + \text{kr3})\ \text{delta}[-\text{kl1} + \text{kr1} + \text{kr2} + \text{kr3}]\ \text{d}[\text{kr3}] ** \text{sigma}[1] ** \text{a}[\text{kr2, s1}] ** \text{Conjugate}[\text{a}[\text{kl1, s2}]] ** \text{b}[\text{kr1}]) \Big/ \\
&\quad \left(16\sqrt{\text{kl1}}\ (\text{kl1} - \text{kr1})\ \sqrt{\text{kr2}}\ (\text{kr2} + \text{kr3})\ \pi\right) + \\
&\quad ((\text{kl1} + \text{kr1} + \text{kr2} - \text{kr3})\ \text{delta}[-\text{kl1} + \text{kr1} + \text{kr2} + \text{kr3}]\ \text{d}[\text{kr3}] ** \text{sigma}[1] ** \text{Conjugate}[\text{a}[\text{kl1, s1}]] ** \text{a}[\text{kr2, s2}] ** \text{b}[\text{kr1}]) \Big/ \\
&\quad \left.\left(16\sqrt{\text{kl1}}\ \sqrt{\text{kr2}}\ (\text{kr1} + \text{kr2})\ (\text{kl1} - \text{kr3})\ \pi\right)\right) + \text{kappaa}\ \text{kappas} \\
&\left(((\text{kl1} + \text{kr1} + \text{kr2} - \text{kr3})\ \text{delta}[-\text{kl1} + \text{kr1} + \text{kr2} + \text{kr3}]\ \text{Conjugate}[\text{b}[\text{kl1}]] ** \text{zeta}[\text{s1}] ** \text{a}[\text{kr3, s1}] ** \text{a}[\text{kr2, s2}] ** \text{b}[\text{kr1}]) \Big/ \right. \\
&\quad \left(16\sqrt{\text{kr2}}\ (\text{kr1} + \text{kr2})\ \sqrt{\text{kr3}}\ (-\text{kl1} + \text{kr3})\ \pi\right) - \\
&\quad ((\text{kl1} + \text{kr1} + \text{kr2} - \text{kr3})\ \text{delta}[-\text{kl1} + \text{kr1} + \text{kr2} + \text{kr3}]\ \text{Conjugate}[\text{b}[\text{kl1}]] ** \text{zeta}[\text{s1}] ** \text{a}[\text{kr3, s2}] ** \text{a}[\text{kr2, s1}] ** \text{b}[\text{kr1}]) \Big/ \\
&\quad \left(16\sqrt{\text{kr2}}\ (\text{kr1} + \text{kr2})\ \sqrt{\text{kr3}}\ (-\text{kl1} + \text{kr3})\ \pi\right) + ((\text{kl1} - \text{kr1} + \text{kr2} + \text{kr3})\ \text{delta}[-\text{kl1} + \text{kr1} + \text{kr2} + \text{kr3}] \\
&\qquad \text{d}[\text{kr3}] ** \text{sigma}[1] ** \text{zeta}[\text{s1}] ** \text{a}[\text{kr2, s1}] ** \text{Conjugate}[\text{a}[\text{kl1, s2}]] ** \text{b}[\text{kr1}]) \Big/ \\
&\quad \left(16\sqrt{\text{kl1}}\ (\text{kl1} - \text{kr1})\ \sqrt{\text{kr2}}\ (\text{kr2} + \text{kr3})\ \pi\right) + ((\text{kl1} - \text{kr1} + \text{kr2} + \text{kr3})\ \text{delta}[-\text{kl1} + \text{kr1} + \text{kr2} + \text{kr3}] \\
&\qquad \text{d}[\text{kr3}] ** \text{sigma}[1] ** \text{zeta}[\text{s1}] ** \text{a}[\text{kr2, s2}] ** \text{Conjugate}[\text{a}[\text{kl1, s1}]] ** \text{b}[\text{kr1}]) \Big/ \\
&\quad \left(16\sqrt{\text{kl1}}\ (\text{kl1} - \text{kr1})\ \sqrt{\text{kr2}}\ (\text{kr2} + \text{kr3})\ \pi\right) + ((\text{kl1} + \text{kr1} + \text{kr2} - \text{kr3})\ \text{delta}[-\text{kl1} + \text{kr1} + \text{kr2} + \text{kr3}] \\
&\qquad \text{d}[\text{kr3}] ** \text{sigma}[1] ** \text{zeta}[\text{s1}] ** \text{Conjugate}[\text{a}[\text{kl1, s1}]] ** \text{a}[\text{kr2, s2}] ** \text{b}[\text{kr1}]) \Big/ \\
&\quad \left(16\sqrt{\text{kl1}}\ \sqrt{\text{kr2}}\ (\text{kr1} + \text{kr2})\ (\text{kl1} - \text{kr3})\ \pi\right) + ((\text{kl1} + \text{kr1} + \text{kr2} - \text{kr3})\ \text{delta}[-\text{kl1} + \text{kr1} + \text{kr2} + \text{kr3}]\ \text{d}[\text{kr3}] ** \\
&\qquad \text{sigma}[1] ** \text{zeta}[\text{s1}] ** \text{Conjugate}[\text{a}[\text{kl1, s2}]] ** \text{a}[\text{kr2, s1}] ** \text{b}[\text{kr1}]) \Big/ \left(16\sqrt{\text{kl1}}\ \sqrt{\text{kr2}}\ (\text{kr1} + \text{kr2})\ (\text{kl1} - \text{kr3})\ \pi\right) - \\
&\quad ((\text{kl1} - \text{kr1} + \text{kr2} + \text{kr3})\ \text{delta}[-\text{kl1} + \text{kr1} + \text{kr2} + \text{kr3}]\ \text{d}[\text{kr3}] ** \text{sigma}[1] ** \text{zeta}[\text{s1}] ** \text{sigma}[1] ** \\
&\qquad \text{a}[\text{kr2, s1}] ** \text{a}[\text{kr1, s2}] ** \text{Conjugate}[\text{d}[\text{kl1}]]) \Big/ \left(16\sqrt{\text{kr1}}\ (-\text{kl1} + \text{kr1})\ \sqrt{\text{kr2}}\ (\text{kr2} + \text{kr3})\ \pi\right) + \\
&\quad ((\text{kl1} - \text{kr1} + \text{kr2} + \text{kr3})\ \text{delta}[-\text{kl1} + \text{kr1} + \text{kr2} + \text{kr3}]\ \text{d}[\text{kr3}] ** \text{sigma}[1] ** \text{zeta}[\text{s1}] ** \text{sigma}[1] ** \\
&\qquad \text{a}[\text{kr2, s2}] ** \text{a}[\text{kr1, s1}] ** \text{Conjugate}[\text{d}[\text{kl1}]]) \Big/ \left.\left(16\sqrt{\text{kr1}}\ (-\text{kl1} + \text{kr1})\ \sqrt{\text{kr2}}\ (\text{kr2} + \text{kr3})\ \pi\right)\right) + \\
&\text{kappaa}^2 \left(((\text{kl1} + \text{kr1} + \text{kr2} - \text{kr3})\ \text{delta}[-\text{kl1} + \text{kr1} + \text{kr2} + \text{kr3}]\ \text{Conjugate}[\text{b}[\text{kl1}]] ** \text{zeta}[\text{s1}] ** \text{zeta}[\text{s2}] ** \right. \\
&\qquad \text{a}[\text{kr3, s1}] ** \text{a}[\text{kr2, s2}] ** \text{b}[\text{kr1}]) \Big/ \left(16\sqrt{\text{kr2}}\ (\text{kr1} + \text{kr2})\ \sqrt{\text{kr3}}\ (-\text{kl1} + \text{kr3})\ \pi\right) - \\
&\quad ((\text{kl1} - \text{kr1} + \text{kr2} + \text{kr3})\ \text{delta}[-\text{kl1} + \text{kr1} + \text{kr2} + \text{kr3}]\ \text{d}[\text{kr3}] ** \text{sigma}[1] ** \text{zeta}[\text{s1}] ** \text{zeta}[\text{s2}] ** \\
&\qquad \text{a}[\text{kr2, s1}] ** \text{Conjugate}[\text{a}[\text{kl1, s2}]] ** \text{b}[\text{kr1}]) \Big/ \left(16\sqrt{\text{kl1}}\ (\text{kl1} - \text{kr1})\ \sqrt{\text{kr2}}\ (\text{kr2} + \text{kr3})\ \pi\right) + \\
&\quad ((\text{kl1} + \text{kr1} + \text{kr2} - \text{kr3})\ \text{delta}[-\text{kl1} + \text{kr1} + \text{kr2} + \text{kr3}]\ \text{d}[\text{kr3}] ** \text{sigma}[1] ** \text{zeta}[\text{s1}] ** \text{zeta}[\text{s2}] ** \\
&\qquad \text{Conjugate}[\text{a}[\text{kl1, s1}]] ** \text{a}[\text{kr2, s2}] ** \text{b}[\text{kr1}]) \Big/ \left(16\sqrt{\text{kl1}}\ \sqrt{\text{kr2}}\ (\text{kr1} + \text{kr2})\ (\text{kl1} - \text{kr3})\ \pi\right) - \\
&\quad ((\text{kl1} - \text{kr1} + \text{kr2} + \text{kr3})\ \text{delta}[-\text{kl1} + \text{kr1} + \text{kr2} + \text{kr3}]\ \text{d}[\text{kr3}] ** \text{sigma}[1] ** \text{zeta}[\text{s1}] ** \text{zeta}[\text{s2}] ** \text{sigma}[1] ** \\
&\qquad \text{a}[\text{kr2, s1}] ** \text{a}[\text{kr1, s2}] ** \text{Conjugate}[\text{d}[\text{kl1}]]) \Big/ \left.\left(16\sqrt{\text{kr1}}\ (-\text{kl1} + \text{kr1})\ \sqrt{\text{kr2}}\ (\text{kr2} + \text{kr3})\ \pi\right)\right)
\end{aligned}
$$

These are the 2→2 particle interactions:

```
expr2 /. a_NonCommutativeMultiply :→
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 2]
```

$$
\begin{aligned}
&\text{kappas}^2 \Big(((\text{kl1} - \text{kl2} + \text{kr1} - \text{kr2})\ \text{delta}[-\text{kl1} - \text{kl2} + \text{kr1} + \text{kr2}]\ \text{Conjugate}[b[\text{kl1}]] ** a[\text{kr2}, \text{s1}] ** \text{Conjugate}[a[\text{kl2}, \text{s2}]] ** b[\text{kr1}]) \Big/ \\
&\quad \Big(16 \sqrt{\text{kl2}}\ (\text{kl2} - \text{kr1})\ \sqrt{\text{kr2}}\ (-\text{kl1} + \text{kr2})\ \pi\Big) + \\
&\quad ((\text{kl1} + \text{kl2} + \text{kr1} + \text{kr2})\ \text{delta}[-\text{kl1} - \text{kl2} + \text{kr1} + \text{kr2}]\ \text{Conjugate}[b[\text{kl1}]] ** \text{Conjugate}[a[\text{kl2}, \text{s1}]] ** a[\text{kr2}, \text{s2}] ** b[\text{kr1}]) \Big/ \\
&\quad \Big(16 \sqrt{\text{kl2}}\ (\text{kl1} + \text{kl2})\ \sqrt{\text{kr2}}\ (\text{kr1} + \text{kr2})\ \pi\Big) - \\
&\quad ((\text{kl1} + \text{kl2} + \text{kr1} + \text{kr2})\ \text{delta}[-\text{kl1} - \text{kl2} + \text{kr1} + \text{kr2}]\ d[\text{kr2}] ** a[\text{kr1}, \text{s1}] ** \text{Conjugate}[a[\text{kl1}, \text{s2}]] ** \text{Conjugate}[d[\text{kl2}]]) \Big/ \\
&\quad \Big(16 \sqrt{\text{kl1}}\ (\text{kl1} + \text{kl2})\ \sqrt{\text{kr1}}\ (\text{kr1} + \text{kr2})\ \pi\Big) + \\
&\quad ((\text{kl1} - \text{kl2} + \text{kr1} - \text{kr2})\ \text{delta}[-\text{kl1} - \text{kl2} + \text{kr1} + \text{kr2}]\ d[\text{kr2}] ** \text{Conjugate}[a[\text{kl1}, \text{s1}]] ** a[\text{kr1}, \text{s2}] ** \text{Conjugate}[d[\text{kl2}]]) \Big/ \\
&\quad \Big(16 \sqrt{\text{kl1}}\ \sqrt{\text{kr1}}\ (-\text{kl2} + \text{kr1})\ (\text{kl1} - \text{kr2})\ \pi\Big) + ((-\text{kl1} + \text{kl2} - \text{kr1} + \text{kr2})\ \text{delta}[-\text{kl1} - \text{kl2} + \text{kr1} + \text{kr2}] \\
&\qquad \text{Conjugate}[b[\text{kl1}]] ** \text{sigma}[1] ** a[\text{kr2}, \text{s1}] ** a[\text{kr1}, \text{s2}] ** \text{Conjugate}[d[\text{kl2}]]) \Big/ \\
&\quad \Big(16\ (\text{kl2} - \text{kr1})\ \sqrt{\text{kr1}}\ (\text{kl1} - \text{kr2})\ \sqrt{\text{kr2}}\ \pi\Big) + ((-\text{kl1} + \text{kl2} - \text{kr1} + \text{kr2})\ \text{delta}[-\text{kl1} - \text{kl2} + \text{kr1} + \text{kr2}]\ d[\text{kr2}] ** \\
&\qquad \text{sigma}[1] ** \text{Conjugate}[a[\text{kl1}, \text{s1}]] ** \text{Conjugate}[a[\text{kl2}, \text{s2}]] ** b[\text{kr1}]) \Big/ \Big(16 \sqrt{\text{kl1}}\ \sqrt{\text{kl2}}\ (\text{kl2} - \text{kr1})\ (\text{kl1} - \text{kr2})\ \pi\Big)\Big) + \\
&\text{kappaa kappas} \Big(((-\text{kl1} + \text{kl2} - \text{kr1} + \text{kr2})\ \text{delta}[-\text{kl1} - \text{kl2} + \text{kr1} + \text{kr2}]\ \text{Conjugate}[b[\text{kl1}]] ** \text{zeta}[\text{s1}] ** \\
&\qquad a[\text{kr2}, \text{s1}] ** \text{Conjugate}[a[\text{kl2}, \text{s2}]] ** b[\text{kr1}]) \Big/ \Big(16 \sqrt{\text{kl2}}\ (\text{kl2} - \text{kr1})\ \sqrt{\text{kr2}}\ (-\text{kl1} + \text{kr2})\ \pi\Big) + \\
&\quad ((-\text{kl1} + \text{kl2} - \text{kr1} + \text{kr2})\ \text{delta}[-\text{kl1} - \text{kl2} + \text{kr1} + \text{kr2}]\ \text{Conjugate}[b[\text{kl1}]] ** \text{zeta}[\text{s1}] ** a[\text{kr2}, \text{s2}] ** \\
&\qquad \text{Conjugate}[a[\text{kl2}, \text{s1}]] ** b[\text{kr1}]) \Big/ \Big(16 \sqrt{\text{kl2}}\ (\text{kl2} - \text{kr1})\ \sqrt{\text{kr2}}\ (-\text{kl1} + \text{kr2})\ \pi\Big) + \\
&\quad ((\text{kl1} + \text{kl2} + \text{kr1} + \text{kr2})\ \text{delta}[-\text{kl1} - \text{kl2} + \text{kr1} + \text{kr2}]\ \text{Conjugate}[b[\text{kl1}]] ** \text{zeta}[\text{s1}] ** \text{Conjugate}[a[\text{kl2}, \text{s1}]] ** \\
&\qquad a[\text{kr2}, \text{s2}] ** b[\text{kr1}]) \Big/ \Big(16 \sqrt{\text{kl2}}\ (\text{kl1} + \text{kl2})\ \sqrt{\text{kr2}}\ (\text{kr1} + \text{kr2})\ \pi\Big) + \\
&\quad ((\text{kl1} + \text{kl2} + \text{kr1} + \text{kr2})\ \text{delta}[-\text{kl1} - \text{kl2} + \text{kr1} + \text{kr2}]\ \text{Conjugate}[b[\text{kl1}]] ** \text{zeta}[\text{s1}] ** \text{Conjugate}[a[\text{kl2}, \text{s2}]] ** \\
&\qquad a[\text{kr2}, \text{s1}] ** b[\text{kr1}]) \Big/ \Big(16 \sqrt{\text{kl2}}\ (\text{kl1} + \text{kl2})\ \sqrt{\text{kr2}}\ (\text{kr1} + \text{kr2})\ \pi\Big) + \\
&\quad ((\text{kl1} - \text{kl2} + \text{kr1} - \text{kr2})\ \text{delta}[-\text{kl1} - \text{kl2} + \text{kr1} + \text{kr2}]\ \text{Conjugate}[b[\text{kl1}]] ** \text{zeta}[\text{s1}] ** \text{sigma}[1] ** \\
&\qquad a[\text{kr2}, \text{s1}] ** a[\text{kr1}, \text{s2}] ** \text{Conjugate}[d[\text{kl2}]]) \Big/ \Big(16\ (\text{kl2} - \text{kr1})\ \sqrt{\text{kr1}}\ (\text{kl1} - \text{kr2})\ \sqrt{\text{kr2}}\ \pi\Big) + \\
&\quad ((-\text{kl1} + \text{kl2} - \text{kr1} + \text{kr2})\ \text{delta}[-\text{kl1} - \text{kl2} + \text{kr1} + \text{kr2}]\ \text{Conjugate}[b[\text{kl1}]] ** \text{zeta}[\text{s1}] ** \text{sigma}[1] ** \\
&\qquad a[\text{kr2}, \text{s2}] ** a[\text{kr1}, \text{s1}] ** \text{Conjugate}[d[\text{kl2}]]) \Big/ \Big(16\ (\text{kl2} - \text{kr1})\ \sqrt{\text{kr1}}\ (\text{kl1} - \text{kr2})\ \sqrt{\text{kr2}}\ \pi\Big) + \\
&\quad ((-\text{kl1} + \text{kl2} - \text{kr1} + \text{kr2})\ \text{delta}[-\text{kl1} - \text{kl2} + \text{kr1} + \text{kr2}]\ d[\text{kr2}] ** \text{sigma}[1] ** \text{zeta}[\text{s1}] ** \text{Conjugate}[a[\text{kl1}, \text{s1}]] ** \\
&\qquad \text{Conjugate}[a[\text{kl2}, \text{s2}]] ** b[\text{kr1}]) \Big/ \Big(16 \sqrt{\text{kl1}}\ \sqrt{\text{kl2}}\ (\text{kl2} - \text{kr1})\ (\text{kl1} - \text{kr2})\ \pi\Big) + \\
&\quad ((\text{kl1} - \text{kl2} + \text{kr1} - \text{kr2})\ \text{delta}[-\text{kl1} - \text{kl2} + \text{kr1} + \text{kr2}]\ d[\text{kr2}] ** \text{sigma}[1] ** \text{zeta}[\text{s1}] ** \text{Conjugate}[a[\text{kl1}, \text{s2}]] ** \\
&\qquad \text{Conjugate}[a[\text{kl2}, \text{s1}]] ** b[\text{kr1}]) \Big/ \Big(16 \sqrt{\text{kl1}}\ \sqrt{\text{kl2}}\ (\text{kl2} - \text{kr1})\ (\text{kl1} - \text{kr2})\ \pi\Big) + \\
&\quad ((\text{kl1} + \text{kl2} + \text{kr1} + \text{kr2})\ \text{delta}[-\text{kl1} - \text{kl2} + \text{kr1} + \text{kr2}]\ d[\text{kr2}] ** \text{sigma}[1] ** \text{zeta}[\text{s1}] ** \text{sigma}[1] ** a[\text{kr1}, \text{s1}] ** \\
&\qquad \text{Conjugate}[a[\text{kl1}, \text{s2}]] ** \text{Conjugate}[d[\text{kl2}]]) \Big/ \Big(16 \sqrt{\text{kl1}}\ (\text{kl1} + \text{kl2})\ \sqrt{\text{kr1}}\ (\text{kr1} + \text{kr2})\ \pi\Big) + \\
&\quad ((\text{kl1} + \text{kl2} + \text{kr1} + \text{kr2})\ \text{delta}[-\text{kl1} - \text{kl2} + \text{kr1} + \text{kr2}]\ d[\text{kr2}] ** \text{sigma}[1] ** \text{zeta}[\text{s1}] ** \text{sigma}[1] ** a[\text{kr1}, \text{s2}] ** \\
&\qquad \text{Conjugate}[a[\text{kl1}, \text{s1}]] ** \text{Conjugate}[d[\text{kl2}]]) \Big/ \Big(16 \sqrt{\text{kl1}}\ (\text{kl1} + \text{kl2})\ \sqrt{\text{kr1}}\ (\text{kr1} + \text{kr2})\ \pi\Big) +
\end{aligned}
$$

```
    ((kl1 - kl2 + kr1 - kr2) delta[-kl1 - kl2 + kr1 + kr2] d[kr2] ** sigma[1] ** zeta[s1] ** sigma[1] **
        Conjugate[a[kl1, s1]] ** a[kr1, s2] ** Conjugate[d[kl2]]) / (16 Sqrt[kl1] Sqrt[kr1] (-kl2 + kr1) (kl1 - kr2) π) +
    ((kl1 - kl2 + kr1 - kr2) delta[-kl1 - kl2 + kr1 + kr2] d[kr2] ** sigma[1] ** zeta[s1] ** sigma[1] **
        Conjugate[a[kl1, s2]] ** a[kr1, s1] ** Conjugate[d[kl2]]) / (16 Sqrt[kl1] Sqrt[kr1] (-kl2 + kr1) (kl1 - kr2) π)) +
kappaa^2 (((kl1 - kl2 + kr1 - kr2) delta[-kl1 - kl2 + kr1 + kr2] Conjugate[b[kl1]] ** zeta[s1] ** zeta[s2] **
        a[kr2, s1] ** Conjugate[a[kl2, s2]] ** b[kr1]) / (16 Sqrt[kl2] (kl2 - kr1) Sqrt[kr2] (-kl1 + kr2) π) +
    ((kl1 + kl2 + kr1 + kr2) delta[-kl1 - kl2 + kr1 + kr2] Conjugate[b[kl1]] ** zeta[s1] ** zeta[s2] **
        Conjugate[a[kl2, s1]] ** a[kr2, s2] ** b[kr1]) / (16 Sqrt[kl2] (kl1 + kl2) Sqrt[kr2] (kr1 + kr2) π) +
    ((kl1 - kl2 + kr1 - kr2) delta[-kl1 - kl2 + kr1 + kr2] Conjugate[b[kl1]] ** zeta[s1] ** zeta[s2] ** sigma[1] **
        a[kr2, s1] ** a[kr1, s2] ** Conjugate[d[kl2]]) / (16 (kl2 - kr1) Sqrt[kr1] (kl1 - kr2) Sqrt[kr2] π) +
    ((kl1 - kl2 + kr1 - kr2) delta[-kl1 - kl2 + kr1 + kr2] d[kr2] ** sigma[1] ** zeta[s1] ** zeta[s2] **
        Conjugate[a[kl1, s1]] ** Conjugate[a[kl2, s2]] ** b[kr1]) / (16 Sqrt[kl1] Sqrt[kl2] (kl2 - kr1) (kl1 - kr2) π) -
    ((kl1 + kl2 + kr1 + kr2) delta[-kl1 - kl2 + kr1 + kr2] d[kr2] ** sigma[1] ** zeta[s1] ** zeta[s2] ** sigma[1] **
        a[kr1, s1] ** Conjugate[a[kl1, s2]] ** Conjugate[d[kl2]]) / (16 Sqrt[kl1] (kl1 + kl2) Sqrt[kr1] (kr1 + kr2) π) +
    ((kl1 - kl2 + kr1 - kr2) delta[-kl1 - kl2 + kr1 + kr2] d[kr2] ** sigma[1] ** zeta[s1] ** zeta[s2] ** sigma[1] **
        Conjugate[a[kl1, s1]] ** a[kr1, s2] ** Conjugate[d[kl2]]) / (16 Sqrt[kl1] Sqrt[kr1] (-kl2 + kr1) (kl1 - kr2) π))
```

These are the 1→3 particle interactions:

```
expr2 /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 3]
```

```
kappas^2
  (-((kl1 + kl2 - kl3 + kr1) delta[-kl1 - kl2 - kl3 + kr1] Conjugate[b[kl1]] ** Conjugate[a[kl2, s1]] ** Conjugate[a[kl3, s2]] **
        b[kr1]) / (16 Sqrt[kl2] (kl1 + kl2) Sqrt[kl3] (kl3 - kr1) π) +
   ((-kl1 + kl2 + kl3 + kr1) delta[-kl1 - kl2 - kl3 + kr1] d[kr1] ** Conjugate[a[kl1, s1]] **
      Conjugate[a[kl2, s2]] ** Conjugate[d[kl3]]) / (16 Sqrt[kl1] Sqrt[kl2] (kl2 + kl3) (kl1 - kr1) π) -
   ((-kl1 + kl2 + kl3 + kr1) delta[-kl1 - kl2 - kl3 + kr1] Conjugate[b[kl1]] ** sigma[1] ** a[kr1, s1] **
      Conjugate[a[kl2, s2]] ** Conjugate[d[kl3]]) / (16 Sqrt[kl2] (kl2 + kl3) Sqrt[kr1] (-kl1 + kr1) π) +
   ((kl1 + kl2 - kl3 + kr1) delta[-kl1 - kl2 - kl3 + kr1] Conjugate[b[kl1]] ** sigma[1] ** Conjugate[a[kl2, s1]] **
      a[kr1, s2] ** Conjugate[d[kl3]]) / (16 Sqrt[kl2] (kl1 + kl2) Sqrt[kr1] (-kl3 + kr1) π)) +
 kappaa kappas (-((kl1 + kl2 - kl3 + kr1) delta[-kl1 - kl2 - kl3 + kr1] Conjugate[b[kl1]] ** zeta[s1] **
        Conjugate[a[kl2, s1]] ** Conjugate[a[kl3, s2]] ** b[kr1]) / (16 Sqrt[kl2] (kl1 + kl2) Sqrt[kl3] (kl3 - kr1) π) +
   ((kl1 + kl2 - kl3 + kr1) delta[-kl1 - kl2 - kl3 + kr1] Conjugate[b[kl1]] ** zeta[s1] ** Conjugate[a[kl2, s2]] **
      Conjugate[a[kl3, s1]] ** b[kr1]) / (16 Sqrt[kl2] (kl1 + kl2) Sqrt[kl3] (kl3 - kr1) π) +
   ((-kl1 + kl2 + kl3 + kr1) delta[-kl1 - kl2 - kl3 + kr1] Conjugate[b[kl1]] ** zeta[s1] ** sigma[1] ** a[kr1, s1] **
      Conjugate[a[kl2, s2]] ** Conjugate[d[kl3]]) / (16 Sqrt[kl2] (kl2 + kl3) Sqrt[kr1] (-kl1 + kr1) π) +
   ((-kl1 + kl2 + kl3 + kr1) delta[-kl1 - kl2 - kl3 + kr1] Conjugate[b[kl1]] ** zeta[s1] ** sigma[1] ** a[kr1, s2] **
      Conjugate[a[kl2, s1]] ** Conjugate[d[kl3]]) / (16 Sqrt[kl2] (kl2 + kl3) Sqrt[kr1] (-kl1 + kr1) π) +
   ((kl1 + kl2 - kl3 + kr1) delta[-kl1 - kl2 - kl3 + kr1] Conjugate[b[kl1]] ** zeta[s1] ** sigma[1] **
      Conjugate[a[kl2, s1]] ** a[kr1, s2] ** Conjugate[d[kl3]]) / (16 Sqrt[kl2] (kl1 + kl2) Sqrt[kr1] (-kl3 + kr1) π) +
   ((kl1 + kl2 - kl3 + kr1) delta[-kl1 - kl2 - kl3 + kr1] Conjugate[b[kl1]] ** zeta[s1] ** sigma[1] **
      Conjugate[a[kl2, s2]] ** a[kr1, s1] ** Conjugate[d[kl3]]) / (16 Sqrt[kl2] (kl1 + kl2) Sqrt[kr1] (-kl3 + kr1) π) +
   ((-kl1 + kl2 + kl3 + kr1) delta[-kl1 - kl2 - kl3 + kr1] d[kr1] ** sigma[1] ** zeta[s1] ** sigma[1] **
      Conjugate[a[kl1, s1]] ** Conjugate[a[kl2, s2]] ** Conjugate[d[kl3]]) / (16 Sqrt[kl1] Sqrt[kl2] (kl2 + kl3) (kl1 - kr1) π) +
   ((kl1 - kl2 - kl3 - kr1) delta[-kl1 - kl2 - kl3 + kr1] d[kr1] ** sigma[1] ** zeta[s1] ** sigma[1] **
      Conjugate[a[kl1, s2]] ** Conjugate[a[kl2, s1]] ** Conjugate[d[kl3]]) / (16 Sqrt[kl1] Sqrt[kl2] (kl2 + kl3) (kl1 - kr1) π)) +
 kappaa^2 (((kl1 + kl2 - kl3 + kr1) delta[-kl1 - kl2 - kl3 + kr1] Conjugate[b[kl1]] ** zeta[s1] ** zeta[s2] **
        Conjugate[a[kl2, s1]] ** Conjugate[a[kl3, s2]] ** b[kr1]) / (16 Sqrt[kl2] (kl1 + kl2) Sqrt[kl3] (kl3 - kr1) π) -
   ((-kl1 + kl2 + kl3 + kr1) delta[-kl1 - kl2 - kl3 + kr1] Conjugate[b[kl1]] ** zeta[s1] ** zeta[s2] ** sigma[1] **
      a[kr1, s1] ** Conjugate[a[kl2, s2]] ** Conjugate[d[kl3]]) / (16 Sqrt[kl2] (kl2 + kl3) Sqrt[kr1] (-kl1 + kr1) π) +
   ((kl1 + kl2 - kl3 + kr1) delta[-kl1 - kl2 - kl3 + kr1] Conjugate[b[kl1]] ** zeta[s1] ** zeta[s2] ** sigma[1] **
      Conjugate[a[kl2, s1]] ** a[kr1, s2] ** Conjugate[d[kl3]]) / (16 Sqrt[kl2] (kl1 + kl2) Sqrt[kr1] (-kl3 + kr1) π) +
   ((kl1 - kl2 - kl3 - kr1) delta[-kl1 - kl2 - kl3 + kr1] d[kr1] ** sigma[1] ** zeta[s1] ** zeta[s2] ** sigma[1] **
      Conjugate[a[kl1, s1]] ** Conjugate[a[kl2, s2]] ** Conjugate[d[kl3]]) / (16 Sqrt[kl1] Sqrt[kl2] (kl2 + kl3) (kl1 - kr1) π))
```

■ Find gauge terms

```
jplus = Expand[I m[x, rr] ** D[m[x, -rr] /. ka → kaa, x] - I D[m[x, rr], x] ** (m[x, -rr] /. ka → kaa) -
        (Conjugate[u[x]] /. {Conjugate[x] → x, Conjugate[kf] → kf}) ** (u[x] /. kf → kff) //. prodexpand /.
      a[b_, -lambda_] ** Conjugate[a[c_, -lambda_]] → a[b, lambda] ** Conjugate[a[c, lambda]] //. prodexpand] //. prodfactor
```

$$\frac{e^{-i\,(ka+kaa)\,x}\,(-ka+kaa)\,a[ka,\,-rr]\,**\,a[kaa,\,rr]}{4\,\sqrt{ka}\,\sqrt{kaa}\,\pi} -$$

$$\frac{e^{-i\,(ka-kaa)\,x}\,(ka+kaa)\,a[ka,\,rr]\,**\,Conjugate[a[kaa,\,rr]]}{4\,\sqrt{ka}\,\sqrt{kaa}\,\pi} + \frac{e^{i\,(ka-kaa)\,x}\,(ka+kaa)\,Conjugate[a[ka,\,rr]]\,**\,a[kaa,\,rr]}{4\,\sqrt{ka}\,\sqrt{kaa}\,\pi} +$$

$$\frac{e^{i\,(ka+kaa)\,x}\,(ka-kaa)\,Conjugate[a[ka,\,rr]]\,**\,Conjugate[a[kaa,\,-rr]]}{4\,\sqrt{ka}\,\sqrt{kaa}\,\pi} -$$

$$\frac{e^{i\,kf\,x-i\,kff\,x}\,Conjugate[b[kf]]\,**\,b[kff]}{2\,\pi} - \frac{e^{-i\,kf\,x+i\,kff\,x}\,d[kf]\,**\,Conjugate[d[kff]]}{2\,\pi} -$$

$$\frac{e^{i\,kf\,x+i\,kff\,x}\,Conjugate[b[kf]]\,**\,sigma[1]\,**\,Conjugate[d[kff]]}{2\,\pi} - \frac{e^{-i\,kf\,x-i\,kff\,x}\,d[kf]\,**\,sigma[1]\,**\,b[kff]}{2\,\pi}$$

I don't use Expand[...] because it makes too many terms.

```
expr3 = g^2 / 4 relabel[(jplus /. {ka → la, kaa → laa, rr → ll, kf → lf, kff → lff}) **
        (I Integrate[I Integrate[jplus, x], x]) //. prodexpand] //. prodfactor;
```

For the gauge interactions, the evaluation of the color indices is implicit and obeys the following rules:
If there are four adjoint fields, the color indices are evaluated as traces:
$a_1\,a_2\,a_3\,a_4 \;\to\; \mathrm{Tr}\{a_1\,a_2\,a_3\,a_4\} - \mathrm{Tr}\{a_1\,a_2\}\,\mathrm{Tr}\{a_3\,a_4\}/N$
If there are two adjoint fields and two fundamental fields, the color indices are evaluated as:
$a_1\,a_2\,b_1\,b_2 \;\to\; b_1\cdot a_1\cdot a_2\cdot b_2 \;-\mathrm{Tr}\{a_1\,a_2\}\;b_1\cdot b_2\,/N$
If there are four fundamental fields, the color indices are evaluated as:
$b\,c\,d\,e \;\to\; b_i\,c_j\,d_k\,e_l\,(\delta_{\mathrm{il}}\,\delta_{\mathrm{kj}} \;-\;\delta_{\mathrm{ij}}\,\delta_{\mathrm{kl}}\,/N)$

```
expr4 = Distribute[int[expr3, {x}] //. intsumexpand /. Exp[x_] :→ Exp[Factor[x]] /. int[Exp[a_ b_], {b_}] → 2 Pi delta[I a]] /.
  delta[_?neverzero] → 0
```

These are the 3→1 particle interactions:

```
expr4 /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 1]
```

$$
\begin{aligned}
&-\left(g^2\ (kl1+kr1)\ (kr2-kr3)\ \text{delta}[-kl1+kr1+kr2+kr3]\ a[kr3,-s1]\ **\ a[kr2,s1]\ **\ a[kr1,s2]\ **\ \text{Conjugate}[a[kl1,s2]]\right)\Big/\\
&\quad\left(32\sqrt{kl1}\ \sqrt{kr1}\ (-kl1+kr1)^2\ \sqrt{kr2}\ \sqrt{kr3}\ \pi\right)+\\
&\left(g^2\ (kl1+kr1)\ (kr2-kr3)\ \text{delta}[-kl1+kr1+kr2+kr3]\ a[kr3,-s1]\ **\ a[kr2,s1]\ **\ \text{Conjugate}[a[kl1,s2]]\ **\ a[kr1,s2]\right)\Big/\\
&\quad\left(32\sqrt{kl1}\ (kl1-kr1)^2\ \sqrt{kr1}\ \sqrt{kr2}\ \sqrt{kr3}\ \pi\right)-\\
&\frac{g^2\ (kr2-kr3)\ \text{delta}[-kl1+kr1+kr2+kr3]\ a[kr3,-s1]\ **\ a[kr2,s1]\ **\ \text{Conjugate}[b[kl1]]\ **\ b[kr1]}{16\ (kl1-kr1)^2\ \sqrt{kr2}\ \sqrt{kr3}\ \pi}-\\
&\frac{g^2\ (kr2-kr3)\ \text{delta}[-kl1+kr1+kr2+kr3]\ a[kr3,-s1]\ **\ a[kr2,s1]\ **\ d[kr1]\ **\ \text{Conjugate}[d[kl1]]}{16\ (-kl1+kr1)^2\ \sqrt{kr2}\ \sqrt{kr3}\ \pi}+\\
&\left(g^2\ (-kr1+kr2)\ (kl1+kr3)\ \text{delta}[-kl1+kr1+kr2+kr3]\ a[kr3,s1]\ **\ \text{Conjugate}[a[kl1,s1]]\ **\ a[kr2,-s2]\ **\ a[kr1,s2]\right)\Big/\\
&\quad\left(32\sqrt{kl1}\ \sqrt{kr1}\ \sqrt{kr2}\ (kr1+kr2)^2\ \sqrt{kr3}\ \pi\right)-\\
&\left(g^2\ (-kr1+kr2)\ (kl1+kr3)\ \text{delta}[-kl1+kr1+kr2+kr3]\ \text{Conjugate}[a[kl1,s1]]\ **\ a[kr3,s1]\ **\ a[kr2,-s2]\ **\ a[kr1,s2]\right)\Big/\\
&\quad\left(32\sqrt{kl1}\ \sqrt{kr1}\ \sqrt{kr2}\ (kr1+kr2)^2\ \sqrt{kr3}\ \pi\right)+\\
&\frac{g^2\ (-kr1+kr2)\ \text{delta}[-kl1+kr1+kr2+kr3]\ \text{Conjugate}[b[kl1]]\ **\ b[kr3]\ **\ a[kr2,-s1]\ **\ a[kr1,s1]}{16\sqrt{kr1}\ \sqrt{kr2}\ (kr1+kr2)^2\ \pi}+\\
&\frac{g^2\ (-kr1+kr2)\ \text{delta}[-kl1+kr1+kr2+kr3]\ d[kr3]\ **\ \text{Conjugate}[d[kl1]]\ **\ a[kr2,-s1]\ **\ a[kr1,s1]}{16\sqrt{kr1}\ \sqrt{kr2}\ (kr1+kr2)^2\ \pi}+\\
&\frac{g^2\ (kl1+kr3)\ \text{delta}[-kl1+kr1+kr2+kr3]\ a[kr3,s1]\ **\ \text{Conjugate}[a[kl1,s1]]\ **\ d[kr2]\ **\ \text{sigma}[1]\ **\ b[kr1]}{16\sqrt{kl1}\ (kr1+kr2)^2\ \sqrt{kr3}\ \pi}-\\
&\frac{g^2\ (kl1+kr3)\ \text{delta}[-kl1+kr1+kr2+kr3]\ \text{Conjugate}[a[kl1,s1]]\ **\ a[kr3,s1]\ **\ d[kr2]\ **\ \text{sigma}[1]\ **\ b[kr1]}{16\sqrt{kl1}\ (kr1+kr2)^2\ \sqrt{kr3}\ \pi}+\\
&\frac{g^2\ \text{delta}[-kl1+kr1+kr2+kr3]\ \text{Conjugate}[b[kl1]]\ **\ b[kr3]\ **\ d[kr2]\ **\ \text{sigma}[1]\ **\ b[kr1]}{8\ (kr1+kr2)^2\ \pi}+\\
&\frac{g^2\ \text{delta}[-kl1+kr1+kr2+kr3]\ d[kr3]\ **\ \text{Conjugate}[d[kl1]]\ **\ d[kr2]\ **\ \text{sigma}[1]\ **\ b[kr1]}{8\ (kr1+kr2)^2\ \pi}+\\
&\frac{g^2\ (kl1+kr1)\ \text{delta}[-kl1+kr1+kr2+kr3]\ d[kr3]\ **\ \text{sigma}[1]\ **\ b[kr2]\ **\ a[kr1,s1]\ **\ \text{Conjugate}[a[kl1,s1]]}{16\sqrt{kl1}\ \sqrt{kr1}\ (-kl1+kr1)^2\ \pi}-\\
&\frac{g^2\ (kl1+kr1)\ \text{delta}[-kl1+kr1+kr2+kr3]\ d[kr3]\ **\ \text{sigma}[1]\ **\ b[kr2]\ **\ \text{Conjugate}[a[kl1,s1]]\ **\ a[kr1,s1]}{16\sqrt{kl1}\ (kl1-kr1)^2\ \sqrt{kr1}\ \pi}+\\
&\frac{g^2\ \text{delta}[-kl1+kr1+kr2+kr3]\ d[kr3]\ **\ \text{sigma}[1]\ **\ b[kr2]\ **\ \text{Conjugate}[b[kl1]]\ **\ b[kr1]}{8\ (kl1-kr1)^2\ \pi}+\\
&\frac{g^2\ \text{delta}[-kl1+kr1+kr2+kr3]\ d[kr3]\ **\ \text{sigma}[1]\ **\ b[kr2]\ **\ d[kr1]\ **\ \text{Conjugate}[d[kl1]]}{8\ (-kl1+kr1)^2\ \pi}
\end{aligned}
$$

These are the 2→2 particle interactions:

```
expr4 /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 2]
```

$$\frac{g^2\,(kl1-kl2)\,(kr1-kr2)\,\delta[-kl1-kl2+kr1+kr2]\;a[kr2,-s1]\star\star a[kr1,s1]\star\star \text{Conjugate}[a[kl1,s2]]\star\star\text{Conjugate}[a[kl2,-s2]]}{32\sqrt{kl1}\,\sqrt{kl2}\,(kl1+kl2)^2\,\sqrt{kr1}\,\sqrt{kr2}\,\pi}+\frac{g^2\,(kl2+kr1)\,(kl1+kr2)\,\delta[-kl1-kl2+kr1+kr2]\;a[kr2,s1]\star\star\text{Conjugate}[a[kl1,s1]]\star\star a[kr1,s2]\star\star\text{Conjugate}[a[kl2,s2]]}{32\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kr1}\,(-kl2+kr1)^2\,\sqrt{kr2}\,\pi}-\frac{g^2\,(kl2+kr1)\,(kl1+kr2)\,\delta[-kl1-kl2+kr1+kr2]\;a[kr2,s1]\star\star\text{Conjugate}[a[kl1,s2]]\star\star\text{Conjugate}[a[kl2,s2]]\star\star a[kr1,s2]}{32\sqrt{kl1}\,\sqrt{kl2}\,(kl2-kr1)^2\,\sqrt{kr1}\,\sqrt{kr2}\,\pi}+\frac{g^2\,(kl1+kr2)\,\delta[-kl1-kl2+kr1+kr2]\;a[kr2,s1]\star\star\text{Conjugate}[a[kl1,s1]]\star\star\text{Conjugate}[b[kl2]]\star\star b[kr1]}{16\sqrt{kl1}\,(kl2-kr1)^2\,\sqrt{kr2}\,\pi}+\frac{g^2\,(kl1+kr2)\,\delta[-kl1-kl2+kr1+kr2]\;a[kr2,s1]\star\star\text{Conjugate}[a[kl1,s1]]\star\star d[kr1]\star\star\text{Conjugate}[d[kl2]]}{16\sqrt{kl1}\,(-kl2+kr1)^2\,\sqrt{kr2}\,\pi}-\frac{g^2\,(kl2+kr1)\,(kl1+kr2)\,\delta[-kl1-kl2+kr1+kr2]\;\text{Conjugate}[a[kl1,s1]]\star\star a[kr2,s1]\star\star a[kr1,s2]\star\star\text{Conjugate}[a[kl2,s2]]}{32\sqrt{kl1}\,\sqrt{kl2}\,(-kl2+kr1)^2\,\sqrt{kr2}\,\pi}+\frac{g^2\,(kl2+kr1)\,(kl1+kr2)\,\delta[-kl1-kl2+kr1+kr2]\;\text{Conjugate}[a[kl1,s1]]\star\star a[kr2,s1]\star\star\text{Conjugate}[a[kl2,s2]]\star\star a[kr1,s2]}{32\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kr1}\,(kl2-kr1)^2\,\sqrt{kr2}\,\pi}-\frac{g^2\,(kl1+kr2)\,\delta[-kl1-kl2+kr1+kr2]\;\text{Conjugate}[a[kl1,s1]]\star\star a[kr2,s1]\star\star\text{Conjugate}[b[kl2]]\star\star b[kr1]}{16\sqrt{kl1}\,(kl2-kr1)^2\,\sqrt{kr2}\,\pi}-\frac{g^2\,(kl1+kr2)\,\delta[-kl1-kl2+kr1+kr2]\;\text{Conjugate}[a[kl1,s1]]\star\star a[kr2,s1]\star\star d[kr1]\star\star\text{Conjugate}[d[kl2]]}{16\sqrt{kl1}\,(-kl2+kr1)^2\,\sqrt{kr2}\,\pi}+\frac{g^2\,(kl1-kl2)\,(-kr1+kr2)\,\delta[-kl1-kl2+kr1+kr2]\;\text{Conjugate}[a[kl1,s1]]\star\star\text{Conjugate}[a[kl2,-s1]]\star\star a[kr2,-s2]\star\star a[kr1,s2]}{32\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kr1}\,\sqrt{kr2}\,(kr1+kr2)^2\,\pi}+\frac{g^2\,(kl2+kr1)\,\delta[-kl1-kl2+kr1+kr2]\;\text{Conjugate}[b[kl1]]\star\star b[kr2]\star\star a[kr1,s1]\star\star\text{Conjugate}[a[kl2,s1]]}{16\sqrt{kl2}\,\sqrt{kr1}\,(-kl2+kr1)^2\,\pi}-\frac{g^2\,(kl2+kr1)\,\delta[-kl1-kl2+kr1+kr2]\;\text{Conjugate}[b[kl1]]\star\star b[kr2]\star\star\text{Conjugate}[a[kl2,s1]]\star\star a[kr1,s1]}{16\sqrt{kl2}\,(kl2-kr1)^2\,\sqrt{kr1}\,\pi}+\frac{g^2\,\delta[-kl1-kl2+kr1+kr2]\;\text{Conjugate}[b[kl1]]\star\star b[kr2]\star\star\text{Conjugate}[b[kl2]]\star\star b[kr1]}{8\,(kl2-kr1)^2\,\pi}+\frac{g^2\,\delta[-kl1-kl2+kr1+kr2]\;\text{Conjugate}[b[kl1]]\star\star b[kr2]\star\star d[kr1]\star\star\text{Conjugate}[d[kl2]]}{8\,(-kl2+kr1)^2\,\pi}+\frac{g^2\,(kl2+kr1)\,\delta[-kl1-kl2+kr1+kr2]\;d[kr2]\star\star\text{Conjugate}[d[kl1]]\star\star a[kr1,s1]\star\star\text{Conjugate}[a[kl2,s1]]}{16\sqrt{kl2}\,\sqrt{kr1}\,(-kl2+kr1)^2\,\pi}-
$$

$$\frac{g^2\,(kl2+kr1)\,\text{delta}[-kl1-kl2+kr1+kr2]\,d[kr2]\star\star\text{Conjugate}[d[kl1]]\star\star\text{Conjugate}[a[kl2,s1]]\star\star a[kr1,s1]}{16\sqrt{kl2}\,(kl2-kr1)^2\sqrt{kr1}\,\pi}+$$

$$\frac{g^2\,\text{delta}[-kl1-kl2+kr1+kr2]\,d[kr2]\star\star\text{Conjugate}[d[kl1]]\star\star\text{Conjugate}[b[kl2]]\star\star b[kr1]}{8\,(kl2-kr1)^2\,\pi}+$$

$$\frac{g^2\,\text{delta}[-kl1-kl2+kr1+kr2]\,d[kr2]\star\star\text{Conjugate}[d[kl1]]\star\star d[kr1]\star\star\text{Conjugate}[d[kl2]]}{8\,(-kl2+kr1)^2\,\pi}-\frac{1}{16\,(kl1+kl2)^2\sqrt{kr1}\,\sqrt{kr2}\,\pi}$$

$$(g^2\,(kr1-kr2)\,\text{delta}[-kl1-kl2+kr1+kr2]\,a[kr2,-s1]\star\star a[kr1,s1]\star\star\text{Conjugate}[b[kl1]]\star\star\text{sigma}[1]\star\star\text{Conjugate}[d[kl2]])-$$

$$\frac{1}{16\sqrt{kl1}\,\sqrt{kl2}\,(kr1+kr2)^2\,\pi}$$

$$(g^2\,(kl1-kl2)\,\text{delta}[-kl1-kl2+kr1+kr2]\,\text{Conjugate}[a[kl1,s1]]\star\star\text{Conjugate}[a[kl2,-s1]]\star\star d[kr2]\star\star\text{sigma}[1]\star\star b[kr1])+$$

$$\frac{1}{16\sqrt{kr1}\,\sqrt{kr2}\,(kr1+kr2)^2\,\pi}\,(g^2\,(-kr1+kr2)\,\text{delta}[-kl1-kl2+kr1+kr2]$$

$$\text{Conjugate}[b[kl1]]\star\star\text{sigma}[1]\star\star\text{Conjugate}[d[kl2]]\star\star a[kr2,-s1]\star\star a[kr1,s1])-\frac{1}{16\sqrt{kl1}\,\sqrt{kl2}\,(kl1+kl2)^2\,\pi}$$

$$(g^2\,(kl1-kl2)\,\text{delta}[-kl1-kl2+kr1+kr2]\,d[kr2]\star\star\text{sigma}[1]\star\star b[kr1]\star\star\text{Conjugate}[a[kl1,s1]]\star\star\text{Conjugate}[a[kl2,-s1]])+$$

$$\frac{g^2\,\text{delta}[-kl1-kl2+kr1+kr2]\,\text{Conjugate}[b[kl1]]\star\star\text{sigma}[1]\star\star\text{Conjugate}[d[kl2]]\star\star d[kr2]\star\star\text{sigma}[1]\star\star b[kr1]}{8\,(kr1+kr2)^2\,\pi}+$$

$$\frac{g^2\,\text{delta}[-kl1-kl2+kr1+kr2]\,d[kr2]\star\star\text{sigma}[1]\star\star b[kr1]\star\star\text{Conjugate}[b[kl1]]\star\star\text{sigma}[1]\star\star\text{Conjugate}[d[kl2]]}{8\,(kl1+kl2)^2\,\pi}$$

These are the 1→3 particle interactions:

```
expr4 /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 3]
```

$$-(g^2\,(kl2-kl3)\,(kl1+kr1)\,\text{delta}[-kl1-kl2-kl3+kr1]$$

$$a[kr1,s1]\star\star\text{Conjugate}[a[kl1,s1]]\star\star\text{Conjugate}[a[kl2,s2]]\star\star\text{Conjugate}[a[kl3,-s2]])\Big/$$

$$\left(32\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,(kl2+kl3)^2\sqrt{kr1}\,\pi\right)+(g^2\,(kl2-kl3)\,(kl1+kr1)\,\text{delta}[-kl1-kl2-kl3+kr1]$$

$$\text{Conjugate}[a[kl1,s1]]\star\star a[kr1,s1]\star\star\text{Conjugate}[a[kl2,s2]]\star\star\text{Conjugate}[a[kl3,-s2]])\Big/$$

$$\left(32\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,(kl2+kl3)^2\sqrt{kr1}\,\pi\right)-(g^2\,(kl1-kl2)\,(kl3+kr1)\,\text{delta}[-kl1-kl2-kl3+kr1]$$

$$\text{Conjugate}[a[kl1,s1]]\star\star\text{Conjugate}[a[kl2,-s1]]\star\star a[kr1,s2]\star\star\text{Conjugate}[a[kl3,s2]])\Big/$$

$$\left(32\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,(-kl3+kr1)^2\,\pi\right)+(g^2\,(kl1-kl2)\,(kl3+kr1)\,\text{delta}[-kl1-kl2-kl3+kr1]$$

$$\text{Conjugate}[a[kl1,s1]]\star\star\text{Conjugate}[a[kl2,-s1]]\star\star\text{Conjugate}[a[kl3,s2]]\star\star a[kr1,s2])\Big/$$

$$\left(32\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,(kl3-kr1)^2\sqrt{kr1}\,\pi\right)-\frac{1}{16\sqrt{kl1}\,\sqrt{kl2}\,(kl3-kr1)^2\,\pi}$$

$$(g^2\,(kl1-kl2)\,\text{delta}[-kl1-kl2-kl3+kr1]\,\text{Conjugate}[a[kl1,s1]]\star\star\text{Conjugate}[a[kl2,-s1]]\star\star\text{Conjugate}[b[kl3]]\star\star b[kr1])-$$

$$(g^2\,(kl1-kl2)\,\text{delta}[-kl1-kl2-kl3+kr1]\,\text{Conjugate}[a[kl1,s1]]\star\star\text{Conjugate}[a[kl2,-s1]]\star\star d[kr1]\star\star\text{Conjugate}[d[kl3]])\Big/$$

$$\left(16\sqrt{\text{kl1}}\ \sqrt{\text{kl2}}\ (-\text{kl3}+\text{kr1})^2\,\pi\right)-\frac{1}{16\sqrt{\text{kl2}}\ \sqrt{\text{kl3}}\ (\text{kl2}+\text{kl3})^2\,\pi}$$
$$(g^2\ (\text{kl2}-\text{kl3})\ \text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]\ \text{Conjugate}[\text{b}[\text{kl1}]]\ **\ \text{b}[\text{kr1}]\ **\ \text{Conjugate}[\text{a}[\text{kl2},\ \text{s1}]]\ **\ \text{Conjugate}[\text{a}[\text{kl3},\ -\text{s1}]])-$$
$$\frac{1}{16\sqrt{\text{kl2}}\ \sqrt{\text{kl3}}\ (\text{kl2}+\text{kl3})^2\,\pi}$$
$$(g^2\ (\text{kl2}-\text{kl3})\ \text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]\ \text{d}[\text{kr1}]\ **\ \text{Conjugate}[\text{d}[\text{kl1}]]\ **\ \text{Conjugate}[\text{a}[\text{kl2},\ \text{s1}]]\ **\ \text{Conjugate}[\text{a}[\text{kl3},\ -\text{s1}]])+$$
$$\frac{1}{16\sqrt{\text{kl1}}\ (\text{kl2}+\text{kl3})^2\sqrt{\text{kr1}}\,\pi}\ (g^2\ (\text{kl1}+\text{kr1})\ \text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]$$
$$\text{a}[\text{kr1},\ \text{s1}]\ **\ \text{Conjugate}[\text{a}[\text{kl1},\ \text{s1}]]\ **\ \text{Conjugate}[\text{b}[\text{kl2}]]\ **\ \text{sigma}[1]\ **\ \text{Conjugate}[\text{d}[\text{kl3}]])-$$
$$\frac{1}{16\sqrt{\text{kl1}}\ (\text{kl2}+\text{kl3})^2\sqrt{\text{kr1}}\,\pi}\ (g^2\ (\text{kl1}+\text{kr1})\ \text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]$$
$$\text{Conjugate}[\text{a}[\text{kl1},\ \text{s1}]]\ **\ \text{a}[\text{kr1},\ \text{s1}]\ **\ \text{Conjugate}[\text{b}[\text{kl2}]]\ **\ \text{sigma}[1]\ **\ \text{Conjugate}[\text{d}[\text{kl3}]])+$$
$$\frac{g^2\ \text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]\ \text{Conjugate}[\text{b}[\text{kl1}]]\ **\ \text{b}[\text{kr1}]\ **\ \text{Conjugate}[\text{b}[\text{kl2}]]\ **\ \text{sigma}[1]\ **\ \text{Conjugate}[\text{d}[\text{kl3}]]}{8\ (\text{kl2}+\text{kl3})^2\,\pi}+$$
$$(g^2\ (\text{kl3}+\text{kr1})\ \text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]$$
$$\text{Conjugate}[\text{b}[\text{kl1}]]\ **\ \text{sigma}[1]\ **\ \text{Conjugate}[\text{d}[\text{kl2}]]\ **\ \text{a}[\text{kr1},\ \text{s1}]\ **\ \text{Conjugate}[\text{a}[\text{kl3},\ \text{s1}]])\ /$$
$$\left(16\sqrt{\text{kl3}}\ \sqrt{\text{kr1}}\ (-\text{kl3}+\text{kr1})^2\,\pi\right)-\frac{1}{16\sqrt{\text{kl3}}\ (\text{kl3}-\text{kr1})^2\sqrt{\text{kr1}}\,\pi}\ (g^2\ (\text{kl3}+\text{kr1})\ \text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]$$
$$\text{Conjugate}[\text{b}[\text{kl1}]]\ **\ \text{sigma}[1]\ **\ \text{Conjugate}[\text{d}[\text{kl2}]]\ **\ \text{Conjugate}[\text{a}[\text{kl3},\ \text{s1}]]\ **\ \text{a}[\text{kr1},\ \text{s1}])+$$
$$\frac{g^2\ \text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]\ \text{Conjugate}[\text{b}[\text{kl1}]]\ **\ \text{sigma}[1]\ **\ \text{Conjugate}[\text{d}[\text{kl2}]]\ **\ \text{Conjugate}[\text{b}[\text{kl3}]]\ **\ \text{b}[\text{kr1}]}{8\ (\text{kl3}-\text{kr1})^2\,\pi}+$$
$$\frac{g^2\ \text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]\ \text{Conjugate}[\text{b}[\text{kl1}]]\ **\ \text{sigma}[1]\ **\ \text{Conjugate}[\text{d}[\text{kl2}]]\ **\ \text{d}[\text{kr1}]\ **\ \text{Conjugate}[\text{d}[\text{kl3}]]}{8\ (-\text{kl3}+\text{kr1})^2\,\pi}+$$
$$\frac{g^2\ \text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]\ \text{d}[\text{kr1}]\ **\ \text{Conjugate}[\text{d}[\text{kl1}]]\ **\ \text{Conjugate}[\text{b}[\text{kl2}]]\ **\ \text{sigma}[1]\ **\ \text{Conjugate}[\text{d}[\text{kl3}]]}{8\ (\text{kl2}+\text{kl3})^2\,\pi}$$

■ Gluon effective potential

I only include the terms that act on meson strings in the large N limit.

```
ka /: Positive[ka] = True; kb /: Positive[kb] = True; kc /: Positive[kc] = True; kd /: Positive[kd] = True;
```

■ The μ_B term

The factor of 1/2 is from the fact that lambda is summing over four values

```
term = mub^2 / 2 m[x, s1] ** (m[x, s1] /. ka → kb);

expr = relabel[
   Expand[term //. prodexpand /. {Conjugate[x] → x, Conjugate[kk_?Positive] → kk, Conjugate[kk_?Positive^n_] → kk^n}] /.
    aa_a ** Conjugate[b_] → Conjugate[b] ** aa];

expr2 = Distribute[int[expr, {x}] //. intsumexpand /. Exp[x_] :→ Exp[Factor[x]] /. int[Exp[a_ b_], {b_}] → 2 Pi delta[I a]] /.
  delta[_?neverzero] → 0
```

$$\frac{\text{mub}^2\ \text{delta}[-\text{kl1}+\text{kr1}]\ \text{Conjugate}[\text{a}[\text{kl1},\ \text{s1}]]\ **\ \text{a}[\text{kr1},\ -\text{s1}]}{2\sqrt{\text{kl1}}\ \sqrt{\text{kr1}}}$$

■ The box term β

The implicit sums are s1={−1,1}and s2={−2,2}. There is an overall factor of 1/4 since there are no sums in the original expression.

```
term = m[x, sa] ** (m[x, sb] /. ka → kb) ** Conjugate[m[x, sa] /. ka → kc] ** Conjugate[m[x, -sb] /. ka → kd];

exprb = relabel[Expand[-beta / 4 (term + Conjugate[term]) //. prodexpand /.
      {Conjugate[x] → x, Conjugate[kk_?Positive] → kk, Conjugate[kk_?Positive^n_] → kk^n}] //.
    {c_ aa_a ** b_ :→ c b ** aa /; MemberQ[b, Conjugate[_]], c_ aa_ ** b_Conjugate :→ c b ** aa /; MemberQ[aa, a[__]]}];

exprbb = Distribute[int[exprb, {x}] //. intsumexpand /. Exp[x_] :→ Exp[Factor[x]] /. int[Exp[a_ b_], {b_}] → 2 Pi delta[I a]] /.
   delta[_?neverzero] → 0;
```

These are the 3→1 particle interactions:

```
exprbb /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 1]
```

$$-\frac{\text{beta}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},-\text{s1}]]\star\star a[\text{kr3},-\text{s2}]\star\star a[\text{kr2},-\text{s1}]\star\star a[\text{kr1},-\text{s2}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},-\text{s1}]]\star\star a[\text{kr3},\text{s2}]\star\star a[\text{kr2},-\text{s1}]\star\star a[\text{kr1},\text{s2}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},\text{s1}]]\star\star a[\text{kr3},-\text{s2}]\star\star a[\text{kr2},\text{s1}]\star\star a[\text{kr1},-\text{s2}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},\text{s1}]]\star\star a[\text{kr3},\text{s2}]\star\star a[\text{kr2},\text{s1}]\star\star a[\text{kr1},\text{s2}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},-\text{s2}]]\star\star a[\text{kr3},-\text{s1}]\star\star a[\text{kr2},\text{s2}]\star\star a[\text{kr1},\text{s1}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},-\text{s2}]]\star\star a[\text{kr3},\text{s1}]\star\star a[\text{kr2},\text{s2}]\star\star a[\text{kr1},-\text{s1}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},\text{s2}]]\star\star a[\text{kr3},-\text{s1}]\star\star a[\text{kr2},-\text{s2}]\star\star a[\text{kr1},\text{s1}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},\text{s2}]]\star\star a[\text{kr3},\text{s1}]\star\star a[\text{kr2},-\text{s2}]\star\star a[\text{kr1},-\text{s1}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi}$$

These are the 2→2 particle interactions:

```
exprbb /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 2]
```

$$-\frac{\text{beta}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-\text{s1}]]\,**\,\text{Conjugate}[a[\text{kl2},-\text{s2}]]\,**\,a[\text{kr2},-\text{s1}]\,**\,a[\text{kr1},\text{s2}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-\text{s1}]]\,**\,\text{Conjugate}[a[\text{kl2},\text{s2}]]\,**\,a[\text{kr2},-\text{s1}]\,**\,a[\text{kr1},-\text{s2}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},\text{s1}]]\,**\,a[\text{kr2},-\text{s2}]\,**\,\text{Conjugate}[a[\text{kl2},-\text{s1}]]\,**\,a[\text{kr1},-\text{s2}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},\text{s1}]]\,**\,a[\text{kr2},\text{s2}]\,**\,\text{Conjugate}[a[\text{kl2},-\text{s1}]]\,**\,a[\text{kr1},\text{s2}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},\text{s1}]]\,**\,\text{Conjugate}[a[\text{kl2},-\text{s2}]]\,**\,a[\text{kr2},\text{s1}]\,**\,a[\text{kr1},\text{s2}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},\text{s1}]]\,**\,\text{Conjugate}[a[\text{kl2},\text{s2}]]\,**\,a[\text{kr2},\text{s1}]\,**\,a[\text{kr1},-\text{s2}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-\text{s2}]]\,**\,a[\text{kr2},-\text{s1}]\,**\,\text{Conjugate}[a[\text{kl2},-\text{s2}]]\,**\,a[\text{kr1},\text{s1}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-\text{s2}]]\,**\,\text{Conjugate}[a[\text{kl2},-\text{s1}]]\,**\,a[\text{kr2},\text{s2}]\,**\,a[\text{kr1},-\text{s1}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-\text{s2}]]\,**\,\text{Conjugate}[a[\text{kl2},\text{s1}]]\,**\,a[\text{kr2},\text{s2}]\,**\,a[\text{kr1},\text{s1}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},\text{s2}]]\,**\,a[\text{kr2},\text{s1}]\,**\,\text{Conjugate}[a[\text{kl2},\text{s2}]]\,**\,a[\text{kr1},-\text{s1}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},\text{s2}]]\,**\,\text{Conjugate}[a[\text{kl2},-\text{s1}]]\,**\,a[\text{kr2},-\text{s2}]\,**\,a[\text{kr1},-\text{s1}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}-$$

$$\frac{\text{beta}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},\text{s2}]]\,**\,\text{Conjugate}[a[\text{kl2},\text{s1}]]\,**\,a[\text{kr2},-\text{s2}]\,**\,a[\text{kr1},\text{s1}]}{32\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}$$

These are the 1→3 particle interactions:

```
exprbb /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 3]
```

$$-\frac{1}{32\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,\pi}$$
(beta delta[-kl1 - kl2 - kl3 + kr1] Conjugate[a[kl1, -s1]] ** Conjugate[a[kl2, -s2]] ** Conjugate[a[kl3, s1]] ** a[kr1, s2]) -
$$\frac{1}{32\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,\pi}$$ (beta delta[-kl1 - kl2 - kl3 + kr1]
Conjugate[a[kl1, -s1]] ** Conjugate[a[kl2, s2]] ** Conjugate[a[kl3, s1]] ** a[kr1, -s2]) - $$\frac{1}{32\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,\pi}$$
(beta delta[-kl1 - kl2 - kl3 + kr1] Conjugate[a[kl1, s1]] ** Conjugate[a[kl2, -s2]] ** Conjugate[a[kl3, -s1]] ** a[kr1, s2]) -
$$\frac{1}{32\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,\pi}$$
(beta delta[-kl1 - kl2 - kl3 + kr1] Conjugate[a[kl1, s1]] ** Conjugate[a[kl2, s2]] ** Conjugate[a[kl3, -s1]] ** a[kr1, -s2]) -
$$\frac{1}{32\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,\pi}$$ (beta delta[-kl1 - kl2 - kl3 + kr1]
Conjugate[a[kl1, -s2]] ** Conjugate[a[kl2, -s1]] ** Conjugate[a[kl3, -s2]] ** a[kr1, -s1]) - $$\frac{1}{32\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,\pi}$$
(beta delta[-kl1 - kl2 - kl3 + kr1] Conjugate[a[kl1, -s2]] ** Conjugate[a[kl2, s1]] ** Conjugate[a[kl3, -s2]] ** a[kr1, s1]) -
$$\frac{1}{32\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,\pi}$$ (beta delta[-kl1 - kl2 - kl3 + kr1]
Conjugate[a[kl1, s2]] ** Conjugate[a[kl2, -s1]] ** Conjugate[a[kl3, s2]] ** a[kr1, -s1]) - $$\frac{1}{32\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,\pi}$$
(beta delta[-kl1 - kl2 - kl3 + kr1] Conjugate[a[kl1, s2]] ** Conjugate[a[kl2, s1]] ** Conjugate[a[kl3, s2]] ** a[kr1, s1])

■ The λ_1 term

There is an implicit sum over the whole expression for s1={1,2,−1,−2}.The factor of 1/2 is because s1 takes on all four values

```
term = lambda1 / 2 m[x, sa] ** Conjugate[m[x, sa] /. ka → kb] ** (m[x, sa] /. ka → kc) ** Conjugate[m[x, sa] /. ka → kd];

exprb = relabel[
   Expand[term //. prodexpand /. {Conjugate[x] → x, Conjugate[kk_?Positive] → kk, Conjugate[kk_?Positive^n_] → kk^n}] //.
    {c_ aa_a ** b_ :> c b ** aa /; MemberQ[b, Conjugate[_]], c_ aa_ ** b_Conjugate :> c b ** aa /; MemberQ[aa, a[__]]}];

exprbb = Distribute[int[exprb, {x}] //. intsumexpand /. Exp[x_] :> Exp[Factor[x]] /. int[Exp[a_ b_], {b_}] → 2 Pi delta[I a]] /.
   delta[_?neverzero] → 0;
```

These are the 3→1 particle interactions:

```
exprbb /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 1]
```

$$\frac{\text{lambda1}\ \text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\ \text{Conjugate}[\text{a}[\text{kl1},-\text{s1}]]\ **\ \text{a}[\text{kr3},-\text{s1}]\ **\ \text{a}[\text{kr2},\text{s1}]\ **\ \text{a}[\text{kr1},-\text{s1}]}{8\sqrt{\text{kl1}}\ \sqrt{\text{kr1}}\ \sqrt{\text{kr2}}\ \sqrt{\text{kr3}}\ \pi} +$$
$$\frac{\text{lambda1}\ \text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\ \text{Conjugate}[\text{a}[\text{kl1},\text{s1}]]\ **\ \text{a}[\text{kr3},\text{s1}]\ **\ \text{a}[\text{kr2},-\text{s1}]\ **\ \text{a}[\text{kr1},\text{s1}]}{8\sqrt{\text{kl1}}\ \sqrt{\text{kr1}}\ \sqrt{\text{kr2}}\ \sqrt{\text{kr3}}\ \pi}$$

These are the 2→2 particle interactions:

```
exprbb /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 2]
```

$$\frac{1}{16\sqrt{\text{kl1}}\ \sqrt{\text{kl2}}\ \sqrt{\text{kr1}}\ \sqrt{\text{kr2}}\ \pi}$$
$$(\text{lambda1}\ \text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\ \text{Conjugate}[\text{a}[\text{kl1},-\text{s1}]]\ **\ \text{a}[\text{kr2},-\text{s1}]\ **\ \text{Conjugate}[\text{a}[\text{kl2},-\text{s1}]]\ **\ \text{a}[\text{kr1},-\text{s1}]) +$$
$$\frac{\text{lambda1}\ \text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\ \text{Conjugate}[\text{a}[\text{kl1},-\text{s1}]]\ **\ \text{Conjugate}[\text{a}[\text{kl2},\text{s1}]]\ **\ \text{a}[\text{kr2},\text{s1}]\ **\ \text{a}[\text{kr1},-\text{s1}]}{8\sqrt{\text{kl1}}\ \sqrt{\text{kl2}}\ \sqrt{\text{kr1}}\ \sqrt{\text{kr2}}\ \pi} +$$
$$\frac{\text{lambda1}\ \text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\ \text{Conjugate}[\text{a}[\text{kl1},\text{s1}]]\ **\ \text{a}[\text{kr2},\text{s1}]\ **\ \text{Conjugate}[\text{a}[\text{kl2},\text{s1}]]\ **\ \text{a}[\text{kr1},\text{s1}]}{16\sqrt{\text{kl1}}\ \sqrt{\text{kl2}}\ \sqrt{\text{kr1}}\ \sqrt{\text{kr2}}\ \pi} +$$
$$\frac{\text{lambda1}\ \text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\ \text{Conjugate}[\text{a}[\text{kl1},\text{s1}]]\ **\ \text{Conjugate}[\text{a}[\text{kl2},-\text{s1}]]\ **\ \text{a}[\text{kr2},-\text{s1}]\ **\ \text{a}[\text{kr1},\text{s1}]}{8\sqrt{\text{kl1}}\ \sqrt{\text{kl2}}\ \sqrt{\text{kr1}}\ \sqrt{\text{kr2}}\ \pi}$$

These are the 1→3 particle interactions:

```
exprbb /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 3]
```

$$\frac{1}{8\sqrt{\text{kl1}}\ \sqrt{\text{kl2}}\ \sqrt{\text{kl3}}\ \sqrt{\text{kr1}}\ \pi}\ (\text{lambda1}\ \text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]$$
$$\text{Conjugate}[\text{a}[\text{kl1},-\text{s1}]]\ **\ \text{Conjugate}[\text{a}[\text{kl2},\text{s1}]]\ **\ \text{Conjugate}[\text{a}[\text{kl3},-\text{s1}]]\ **\ \text{a}[\text{kr1},-\text{s1}]) + \frac{1}{8\sqrt{\text{kl1}}\ \sqrt{\text{kl2}}\ \sqrt{\text{kl3}}\ \sqrt{\text{kr1}}\ \pi}$$
$$(\text{lambda1}\ \text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]\ \text{Conjugate}[\text{a}[\text{kl1},\text{s1}]]\ **\ \text{Conjugate}[\text{a}[\text{kl2},-\text{s1}]]\ **\ \text{Conjugate}[\text{a}[\text{kl3},\text{s1}]]\ **\ \text{a}[\text{kr1},\text{s1}])$$

■ The λ_2 term

There is an implicit sum over the whole expression for s1={1,2,–1,–2}.The factor of 1/2 is because s1 takes on all four values

```
term = lambda2 / 2 m[x, sa] ** (m[x, sa] /. ka → kb) ** Conjugate[m[x, sa] /. ka → kc] ** Conjugate[m[x, sa] /. ka → kd];
```

```
exprb = relabel[
   Expand[term //. prodexpand /. {Conjugate[x] → x, Conjugate[kk_?Positive] → kk, Conjugate[kk_?Positive^n_] → kk^n}] //.
    {c_ aa_a ** b_ :→ c b ** aa /; MemberQ[b, Conjugate[_]], c_ aa_ ** b_Conjugate :→ c b ** aa /; MemberQ[aa, a[__]]}];

exprbb = Distribute[int[exprb, {x}] //. intsumexpand /. Exp[x_] :→ Exp[Factor[x]] /. int[Exp[a_ b_], {b_}] → 2 Pi delta[I a]] /.
   delta[_?neverzero] → 0;
```

These are the 3→1 particle interactions:

```
exprbb /. a_NonCommutativeMultiply :→
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 1]
```

$$\frac{\text{lambda2}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},-\text{s1}]]\star\star a[\text{kr3},-\text{s1}]\star\star a[\text{kr2},-\text{s1}]\star\star a[\text{kr1},\text{s1}]}{16\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi}+$$
$$\frac{\text{lambda2}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},-\text{s1}]]\star\star a[\text{kr3},\text{s1}]\star\star a[\text{kr2},-\text{s1}]\star\star a[\text{kr1},-\text{s1}]}{16\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi}+$$
$$\frac{\text{lambda2}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},\text{s1}]]\star\star a[\text{kr3},-\text{s1}]\star\star a[\text{kr2},\text{s1}]\star\star a[\text{kr1},\text{s1}]}{16\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi}+$$
$$\frac{\text{lambda2}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},\text{s1}]]\star\star a[\text{kr3},\text{s1}]\star\star a[\text{kr2},\text{s1}]\star\star a[\text{kr1},-\text{s1}]}{16\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi}$$

These are the 2→2 particle interactions:

```
exprbb /. a_NonCommutativeMultiply :→
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 2]
```

$$\frac{1}{16\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}$$
$$(\text{lambda2}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-\text{s1}]]\star\star\text{Conjugate}[a[\text{kl2},-\text{s1}]]\star\star a[\text{kr2},-\text{s1}]\star\star a[\text{kr1},-\text{s1}])+$$
$$\frac{\text{lambda2}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-\text{s1}]]\star\star\text{Conjugate}[a[\text{kl2},\text{s1}]]\star\star a[\text{kr2},-\text{s1}]\star\star a[\text{kr1},\text{s1}]}{16\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}+$$
$$\frac{\text{lambda2}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},\text{s1}]]\star\star a[\text{kr2},-\text{s1}]\star\star\text{Conjugate}[a[\text{kl2},-\text{s1}]]\star\star a[\text{kr1},\text{s1}]}{16\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}+$$
$$\frac{\text{lambda2}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},\text{s1}]]\star\star a[\text{kr2},\text{s1}]\star\star\text{Conjugate}[a[\text{kl2},-\text{s1}]]\star\star a[\text{kr1},-\text{s1}]}{16\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}+$$
$$\frac{\text{lambda2}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},\text{s1}]]\star\star\text{Conjugate}[a[\text{kl2},-\text{s1}]]\star\star a[\text{kr2},\text{s1}]\star\star a[\text{kr1},-\text{s1}]}{16\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}+$$
$$\frac{\text{lambda2}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},\text{s1}]]\star\star\text{Conjugate}[a[\text{kl2},\text{s1}]]\star\star a[\text{kr2},\text{s1}]\star\star a[\text{kr1},\text{s1}]}{16\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}$$

These are the 1→3 particle interactions:

```
exprbb /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 3]
```

$$\frac{1}{16\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,\pi}\,(\text{lambda2}\,\text{delta}[-kl1-kl2-kl3+kr1]$$
$$\text{Conjugate}[a[kl1,-s1]]\,**\,\text{Conjugate}[a[kl2,-s1]]\,**\,\text{Conjugate}[a[kl3,s1]]\,**\,a[kr1,-s1])+$$
$$\frac{1}{16\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,\pi}\,(\text{lambda2}\,\text{delta}[-kl1-kl2-kl3+kr1]$$
$$\text{Conjugate}[a[kl1,-s1]]\,**\,\text{Conjugate}[a[kl2,s1]]\,**\,\text{Conjugate}[a[kl3,s1]]\,**\,a[kr1,s1])+$$
$$\frac{1}{16\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,\pi}\,(\text{lambda2}\,\text{delta}[-kl1-kl2-kl3+kr1]$$
$$\text{Conjugate}[a[kl1,s1]]\,**\,\text{Conjugate}[a[kl2,-s1]]\,**\,\text{Conjugate}[a[kl3,-s1]]\,**\,a[kr1,-s1])+\frac{1}{16\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,\pi}$$
$$(\text{lambda2}\,\text{delta}[-kl1-kl2-kl3+kr1]\,\text{Conjugate}[a[kl1,s1]]\,**\,\text{Conjugate}[a[kl2,s1]]\,**\,\text{Conjugate}[a[kl3,-s1]]\,**\,a[kr1,s1])$$

■ The λ_4 term

The implicit sums are s1={−1,1}and s2={−2,2}.

```
term = lambda4 m[x, sa] ** Conjugate[m[x, sa] /. ka → kb] ** (m[x, sb] /. ka → kc) ** Conjugate[m[x, sb] /. ka → kd];

exprb = relabel[
   Expand[term //. prodexpand /. {Conjugate[x] → x, Conjugate[kk_?Positive] → kk, Conjugate[kk_?Positive^n_] → kk^n}] //.
    {c_ aa_a ** bb_ :> c bb ** aa /; MemberQ[bb, Conjugate[_]], c_ aa_ ** bb_Conjugate :> c bb ** aa /; MemberQ[aa, a[__]]}];

exprbb = Distribute[int[exprb, {x}] //. intsumexpand /. Exp[x_] :> Exp[Factor[x]] /. int[Exp[a_ b_], {b_}] → 2 Pi delta[I a]] /.
   delta[_?neverzero] → 0;
```

These are the 3→1 particle interactions:

```
exprbb /. a_NonCommutativeMultiply :→
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 1]
```

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},-\text{s1}]]\star\star a[\text{kr3},-\text{s2}]\star\star a[\text{kr2},\text{s2}]\star\star a[\text{kr1},-\text{s1}]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$
$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},\text{s1}]]\star\star a[\text{kr3},\text{s1}]\star\star a[\text{kr2},-\text{s2}]\star\star a[\text{kr1},\text{s2}]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$
$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},-\text{s2}]]\star\star a[\text{kr3},-\text{s1}]\star\star a[\text{kr2},\text{s1}]\star\star a[\text{kr1},-\text{s2}]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$
$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},\text{s2}]]\star\star a[\text{kr3},\text{s2}]\star\star a[\text{kr2},-\text{s1}]\star\star a[\text{kr1},\text{s1}]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi}$$

These are the 2→2 particle interactions:

```
exprbb /. a_NonCommutativeMultiply :→
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 2]
```

$$\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}$$
$$(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-\text{s1}]]\star\star a[\text{kr2},-\text{s2}]\star\star\text{Conjugate}[a[\text{kl2},-\text{s2}]]\star\star a[\text{kr1},-\text{s1}]) +$$
$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-\text{s1}]]\star\star\text{Conjugate}[a[\text{kl2},\text{s2}]]\star\star a[\text{kr2},\text{s2}]\star\star a[\text{kr1},-\text{s1}]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$
$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},\text{s1}]]\star\star a[\text{kr2},\text{s1}]\star\star\text{Conjugate}[a[\text{kl2},\text{s2}]]\star\star a[\text{kr1},\text{s2}]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$
$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},\text{s1}]]\star\star\text{Conjugate}[a[\text{kl2},-\text{s1}]]\star\star a[\text{kr2},-\text{s2}]\star\star a[\text{kr1},\text{s2}]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$
$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-\text{s2}]]\star\star\text{Conjugate}[a[\text{kl2},\text{s1}]]\star\star a[\text{kr2},\text{s1}]\star\star a[\text{kr1},-\text{s2}]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$
$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},\text{s2}]]\star\star\text{Conjugate}[a[\text{kl2},-\text{s2}]]\star\star a[\text{kr2},-\text{s1}]\star\star a[\text{kr1},\text{s1}]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}$$

These are the 1→3 particle interactions:

```
exprbb /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 3]
```

$$\frac{1}{8\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,\pi}\,(\text{lambda4}\,\text{delta}[-kl1-kl2-kl3+kr1]$$
$$\text{Conjugate}[a[kl1,-s1]] ** \text{Conjugate}[a[kl2,s2]] ** \text{Conjugate}[a[kl3,-s2]] ** a[kr1,-s1]) +$$
$$\frac{1}{8\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,\pi}\,(\text{lambda4}\,\text{delta}[-kl1-kl2-kl3+kr1]$$
$$\text{Conjugate}[a[kl1,s1]] ** \text{Conjugate}[a[kl2,-s1]] ** \text{Conjugate}[a[kl3,s2]] ** a[kr1,s2]) +$$
$$\frac{1}{8\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,\pi}\,(\text{lambda4}\,\text{delta}[-kl1-kl2-kl3+kr1]$$
$$\text{Conjugate}[a[kl1,-s2]] ** \text{Conjugate}[a[kl2,s1]] ** \text{Conjugate}[a[kl3,-s1]] ** a[kr1,-s2]) + \frac{1}{8\sqrt{kl1}\,\sqrt{kl2}\,\sqrt{kl3}\,\sqrt{kr1}\,\pi}$$
$$(\text{lambda4}\,\text{delta}[-kl1-kl2-kl3+kr1]\,\text{Conjugate}[a[kl1,s2]] ** \text{Conjugate}[a[kl2,-s2]] ** \text{Conjugate}[a[kl3,s1]] ** a[kr1,s1])$$

■ The λ_4 term (done more explicitly)

The implicit sums are s1={−1,1}and s2={−2,2}.

```
term = lambda4 Sum[m[x, sa] ** Conjugate[m[x, sa] /. ka → kb] **
     (m[x, sb] /. ka → kc) ** Conjugate[m[x, sb] /. ka → kd], {sa, -1, 1, 2}, {sb, -2, 2, 4}];

exprb = relabel[
    Expand[term //. prodexpand /. {Conjugate[x] → x, Conjugate[kk_?Positive] → kk, Conjugate[kk_?Positive^n_] → kk^n}] //.
     {c_ aa_a ** bb_ :> c bb ** aa /; MemberQ[bb, Conjugate[_]],
      c_ aa_ ** bb_Conjugate :> c bb ** aa /; MemberQ[aa, a[__]]}] //. prodfactor;

exprbb = Distribute[int[exprb, {x}] //. intsumexpand /. Exp[x_] :> Exp[Factor[x]] /. int[Exp[a_ b_], {b_}] → 2 Pi delta[I a]] /.
   delta[_?neverzero] → 0;
```

These are the 3→1 particle interactions:

```
exprbb /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 1]
```

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},-2]]\star\star a[\text{kr3},-2]\star\star a[\text{kr2},-1]\star\star a[\text{kr1},1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},-2]]\star\star a[\text{kr3},-2]\star\star a[\text{kr2},1]\star\star a[\text{kr1},-1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},-2]]\star\star a[\text{kr3},-1]\star\star a[\text{kr2},1]\star\star a[\text{kr1},-2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},-2]]\star\star a[\text{kr3},1]\star\star a[\text{kr2},-1]\star\star a[\text{kr1},-2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},-1]]\star\star a[\text{kr3},-2]\star\star a[\text{kr2},2]\star\star a[\text{kr1},-1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},-1]]\star\star a[\text{kr3},-1]\star\star a[\text{kr2},-2]\star\star a[\text{kr1},2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},-1]]\star\star a[\text{kr3},-1]\star\star a[\text{kr2},2]\star\star a[\text{kr1},-2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},-1]]\star\star a[\text{kr3},2]\star\star a[\text{kr2},-2]\star\star a[\text{kr1},-1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},1]]\star\star a[\text{kr3},-2]\star\star a[\text{kr2},2]\star\star a[\text{kr1},1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},1]]\star\star a[\text{kr3},1]\star\star a[\text{kr2},-2]\star\star a[\text{kr1},2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},1]]\star\star a[\text{kr3},1]\star\star a[\text{kr2},2]\star\star a[\text{kr1},-2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},1]]\star\star a[\text{kr3},2]\star\star a[\text{kr2},-2]\star\star a[\text{kr1},1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},2]]\star\star a[\text{kr3},-1]\star\star a[\text{kr2},1]\star\star a[\text{kr1},2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},2]]\star\star a[\text{kr3},1]\star\star a[\text{kr2},-1]\star\star a[\text{kr1},2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},2]]\star\star a[\text{kr3},2]\star\star a[\text{kr2},-1]\star\star a[\text{kr1},1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}+\text{kr1}+\text{kr2}+\text{kr3}]\,\text{Conjugate}[a[\text{kl1},2]]\star\star a[\text{kr3},2]\star\star a[\text{kr2},1]\star\star a[\text{kr1},-1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\sqrt{\text{kr3}}\,\pi}$$

These are the 2→2 particle interactions:

```
exprbb /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 2]
```

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-2]]\,**\,\text{Conjugate}[a[\text{kl2},-1]]\,**\,a[\text{kr2},-1]\,**\,a[\text{kr1},-2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-2]]\,**\,\text{Conjugate}[a[\text{kl2},1]]\,**\,a[\text{kr2},1]\,**\,a[\text{kr1},-2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-2]]\,**\,\text{Conjugate}[a[\text{kl2},2]]\,**\,a[\text{kr2},-1]\,**\,a[\text{kr1},1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-2]]\,**\,\text{Conjugate}[a[\text{kl2},2]]\,**\,a[\text{kr2},1]\,**\,a[\text{kr1},-1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-1]]\,**\,a[\text{kr2},-2]\,**\,\text{Conjugate}[a[\text{kl2},-2]]\,**\,a[\text{kr1},-1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-1]]\,**\,a[\text{kr2},-1]\,**\,\text{Conjugate}[a[\text{kl2},-2]]\,**\,a[\text{kr1},-2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-1]]\,**\,a[\text{kr2},-1]\,**\,\text{Conjugate}[a[\text{kl2},2]]\,**\,a[\text{kr1},2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-1]]\,**\,a[\text{kr2},2]\,**\,\text{Conjugate}[a[\text{kl2},2]]\,**\,a[\text{kr1},-1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-1]]\,**\,\text{Conjugate}[a[\text{kl2},-2]]\,**\,a[\text{kr2},-2]\,**\,a[\text{kr1},-1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-1]]\,**\,\text{Conjugate}[a[\text{kl2},1]]\,**\,a[\text{kr2},-2]\,**\,a[\text{kr1},2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-1]]\,**\,\text{Conjugate}[a[\text{kl2},1]]\,**\,a[\text{kr2},2]\,**\,a[\text{kr1},-2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},-1]]\,**\,\text{Conjugate}[a[\text{kl2},2]]\,**\,a[\text{kr2},2]\,**\,a[\text{kr1},-1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},1]]\,**\,a[\text{kr2},-2]\,**\,\text{Conjugate}[a[\text{kl2},-2]]\,**\,a[\text{kr1},1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},1]]\,**\,a[\text{kr2},1]\,**\,\text{Conjugate}[a[\text{kl2},-2]]\,**\,a[\text{kr1},-2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},1]]\,**\,a[\text{kr2},1]\,**\,\text{Conjugate}[a[\text{kl2},2]]\,**\,a[\text{kr1},2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},1]]\,**\,a[\text{kr2},2]\,**\,\text{Conjugate}[a[\text{kl2},2]]\,**\,a[\text{kr1},1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},1]]\,**\,\text{Conjugate}[a[\text{kl2},-2]]\,**\,a[\text{kr2},-2]\,**\,a[\text{kr1},1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi} +$$

$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},1]]\star\star\text{Conjugate}[a[\text{kl2},-1]]\star\star a[\text{kr2},-2]\star\star a[\text{kr1},2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}+$$
$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},1]]\star\star\text{Conjugate}[a[\text{kl2},-1]]\star\star a[\text{kr2},2]\star\star a[\text{kr1},-2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}+$$
$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},1]]\star\star\text{Conjugate}[a[\text{kl2},2]]\star\star a[\text{kr2},2]\star\star a[\text{kr1},1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}+$$
$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},2]]\star\star\text{Conjugate}[a[\text{kl2},-2]]\star\star a[\text{kr2},-1]\star\star a[\text{kr1},1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}+$$
$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},2]]\star\star\text{Conjugate}[a[\text{kl2},-2]]\star\star a[\text{kr2},1]\star\star a[\text{kr1},-1]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}+$$
$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},2]]\star\star\text{Conjugate}[a[\text{kl2},-1]]\star\star a[\text{kr2},-1]\star\star a[\text{kr1},2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}+$$
$$\frac{\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}+\text{kr1}+\text{kr2}]\,\text{Conjugate}[a[\text{kl1},2]]\star\star\text{Conjugate}[a[\text{kl2},1]]\star\star a[\text{kr2},1]\star\star a[\text{kr1},2]}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kr1}}\,\sqrt{\text{kr2}}\,\pi}$$

These are the 1→3 particle interactions:

```
exprbb /. a_NonCommutativeMultiply :>
  Block[{non = a //. {sigma[_] ** c_ → c, zeta[_] ** c_ → c}}, 0 /; Length[non] ≠ 4 || Count[non, Conjugate[_]] ≠ 3]
```

$$\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kl3}}\,\sqrt{\text{kr1}}\,\pi}$$
$$(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]\,\text{Conjugate}[a[\text{kl1},-2]]\star\star\text{Conjugate}[a[\text{kl2},-1]]\star\star\text{Conjugate}[a[\text{kl3},1]]\star\star a[\text{kr1},-2])+$$
$$\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kl3}}\,\sqrt{\text{kr1}}\,\pi}$$
$$(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]\,\text{Conjugate}[a[\text{kl1},-2]]\star\star\text{Conjugate}[a[\text{kl2},1]]\star\star\text{Conjugate}[a[\text{kl3},-1]]\star\star a[\text{kr1},-2])+$$
$$\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kl3}}\,\sqrt{\text{kr1}}\,\pi}(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]$$
$$\text{Conjugate}[a[\text{kl1},-2]]\star\star\text{Conjugate}[a[\text{kl2},2]]\star\star\text{Conjugate}[a[\text{kl3},-1]]\star\star a[\text{kr1},-1])+\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kl3}}\,\sqrt{\text{kr1}}\,\pi}$$
$$(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]\,\text{Conjugate}[a[\text{kl1},-2]]\star\star\text{Conjugate}[a[\text{kl2},2]]\star\star\text{Conjugate}[a[\text{kl3},1]]\star\star a[\text{kr1},1])+$$
$$\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kl3}}\,\sqrt{\text{kr1}}\,\pi}(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]$$
$$\text{Conjugate}[a[\text{kl1},-1]]\star\star\text{Conjugate}[a[\text{kl2},-2]]\star\star\text{Conjugate}[a[\text{kl3},2]]\star\star a[\text{kr1},-1])+\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kl3}}\,\sqrt{\text{kr1}}\,\pi}$$
$$(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]\,\text{Conjugate}[a[\text{kl1},-1]]\star\star\text{Conjugate}[a[\text{kl2},1]]\star\star\text{Conjugate}[a[\text{kl3},-2]]\star\star a[\text{kr1},-2])+$$
$$\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kl3}}\,\sqrt{\text{kr1}}\,\pi}(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]$$
$$\text{Conjugate}[a[\text{kl1},-1]]\star\star\text{Conjugate}[a[\text{kl2},1]]\star\star\text{Conjugate}[a[\text{kl3},2]]\star\star a[\text{kr1},2])+\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kl3}}\,\sqrt{\text{kr1}}\,\pi}$$

$$(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]\,\text{Conjugate}[a[\text{kl1},-1]]\star\star\text{Conjugate}[a[\text{kl2},2]]\star\star\text{Conjugate}[a[\text{kl3},-2]]\star\star a[\text{kr1},-1])+$$
$$\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kl3}}\,\sqrt{\text{kr1}}\,\pi}(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]$$
$$\text{Conjugate}[a[\text{kl1},1]]\star\star\text{Conjugate}[a[\text{kl2},-2]]\star\star\text{Conjugate}[a[\text{kl3},2]]\star\star a[\text{kr1},1])+\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kl3}}\,\sqrt{\text{kr1}}\,\pi}$$
$$(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]\,\text{Conjugate}[a[\text{kl1},1]]\star\star\text{Conjugate}[a[\text{kl2},-1]]\star\star\text{Conjugate}[a[\text{kl3},-2]]\star\star a[\text{kr1},-2])+$$
$$\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kl3}}\,\sqrt{\text{kr1}}\,\pi}(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]$$
$$\text{Conjugate}[a[\text{kl1},1]]\star\star\text{Conjugate}[a[\text{kl2},-1]]\star\star\text{Conjugate}[a[\text{kl3},2]]\star\star a[\text{kr1},2])+\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kl3}}\,\sqrt{\text{kr1}}\,\pi}$$
$$(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]\,\text{Conjugate}[a[\text{kl1},1]]\star\star\text{Conjugate}[a[\text{kl2},2]]\star\star\text{Conjugate}[a[\text{kl3},-2]]\star\star a[\text{kr1},1])+$$
$$\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kl3}}\,\sqrt{\text{kr1}}\,\pi}(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]$$
$$\text{Conjugate}[a[\text{kl1},2]]\star\star\text{Conjugate}[a[\text{kl2},-2]]\star\star\text{Conjugate}[a[\text{kl3},-1]]\star\star a[\text{kr1},-1])+\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kl3}}\,\sqrt{\text{kr1}}\,\pi}$$
$$(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]\,\text{Conjugate}[a[\text{kl1},2]]\star\star\text{Conjugate}[a[\text{kl2},-2]]\star\star\text{Conjugate}[a[\text{kl3},1]]\star\star a[\text{kr1},1])+$$
$$\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kl3}}\,\sqrt{\text{kr1}}\,\pi}(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]$$
$$\text{Conjugate}[a[\text{kl1},2]]\star\star\text{Conjugate}[a[\text{kl2},-1]]\star\star\text{Conjugate}[a[\text{kl3},1]]\star\star a[\text{kr1},2])+\frac{1}{8\sqrt{\text{kl1}}\,\sqrt{\text{kl2}}\,\sqrt{\text{kl3}}\,\sqrt{\text{kr1}}\,\pi}$$
$$(\text{lambda4}\,\text{delta}[-\text{kl1}-\text{kl2}-\text{kl3}+\text{kr1}]\,\text{Conjugate}[a[\text{kl1},2]]\star\star\text{Conjugate}[a[\text{kl2},1]]\star\star\text{Conjugate}[a[\text{kl3},-1]]\star\star a[\text{kr1},2])$$

■ Junk