Dear Simon,

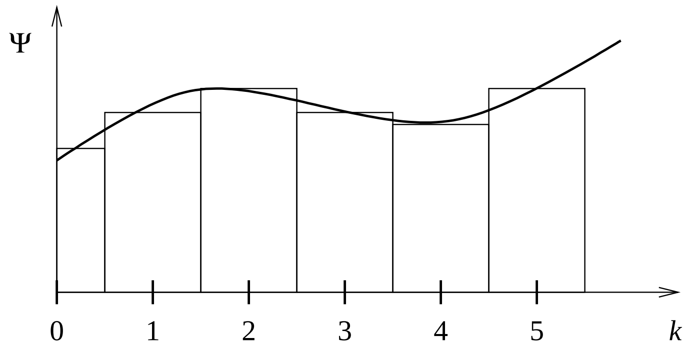
Here are some notes on the solving the zero mode problem for the link fields. The following discussion concentrates on the DLCQ case. I have some ideas of doing this for "improved" matrix elements, but I haven't written anything out, yet.

The overall philosophy of this approach is to take the bound state integral equations to be the "definition" of the theory. Our use of DLCQ is rather pragmatic: it is simply a numerical technique for approximating the integral equations. The zero mode represents the zero momentum limit of the continuum wavefunction.

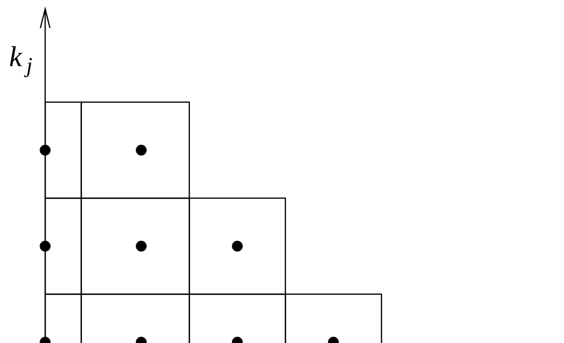
In Section 2, we use a constant background field to produce the desired bound state equations. This approach is a bit of overkill for the problem at hand. Moreover, I haven't worked out its generalization to the case of more than two links. However, it makes for a good introduction to the concepts.

1 Phase space

The first thing that you notice about the zero mode in the case of the two particle bound state, is that it represents half the phase space that an ordinary (nonzero) mode does:



Similarly, when two momenta go to zero, the phase space is one quarter that of an ordinary mode.



In general, a state with p zero modes occupies 2^{-p} as much phase space as an ordinary nonzero momentum mode. It is interesting to note that the constrained zero modes, as computed with the Dirac-Bergmann formalism, produces the correct factor of 2^{-p}.

2 Background field

Consider a transverse lattice with periodic boundary conditions in the x^--direction. Assume that A_- is a constant Abelian background field. Then the longitudinal gauge term in the Lagrangian is

$$-\frac{1}{2g^2}\mathrm{Tr}\left\{F_{\alpha\beta}F^{\alpha\beta}\right\} = \frac{N}{g^2}\left(\partial_+ A_-\right)^2 + \frac{1}{g^2}\mathrm{Tr}\left\{\left(\partial_- A_+\right)^2\right\} \tag{1}$$

so A_- is an ordinary dynamical field. (See Coleman's paper on the Schwinger model θ-vacuum.) The covariant derivative is

$$D_- M_r(\mathbf{x}) = \partial_- M_r(\mathbf{x}) + iA_-(\mathbf{x})M_r(\mathbf{x}) - iM_r(\mathbf{x})A_-(\mathbf{x}+\hat{r}) \ . \tag{2}$$

We see that if A_- is the same at every site, there is no effect on the link fields. However, if A_- is different on two neighboring sites, the momentum of the link field is shifted by a constant. The momenta of the link M_r and the anti-link $M_r^\dagger$ are shifted in opposite directions. Although A_- is a dynamical field, we will choose a wavefunction for it. Define eigenstates of A_- at site $\mathbf{x}$:

$$A_- |\mathbf{x},\pm\rangle = \pm\frac{\pi\epsilon}{L}|\mathbf{x},\pm\rangle \tag{3}$$

We will define the vacuum on the transverse lattice to be a superposition of these two states in a checkerboard fashion. In particular, if we have just two sites:

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|\mathbf{x},+\rangle\,|\mathbf{x}+\hat{r},-\rangle + |\mathbf{x},-\rangle\,|\mathbf{x}+\hat{r},+\rangle\right) \ . \tag{4}$$

Given this vacuum, we can quantize the M_r fields. The relevant term in the Lagrangian is

$$\int_0^L dx^-\,\mathrm{Tr}\left\{\partial_+ M_r\left(D_- M_r\right)^\dagger + \partial_+ M_r^\dagger\left(D_- M_r\right)\right\} \ . \tag{5}$$

In order to obtain the ordinary commutation relations for the creation-annihilation operators $a_r(k)$ and $a_r(k)^\dagger$, we choose

$$M_r(x^-,\mathbf{x}) = \frac{1}{\sqrt{4\pi}L}\sum_{k=0}^{\infty}\left(\frac{a_{-r}(k,\mathbf{x})\,e^{-2\pi i x^- k/L}}{\sqrt{k-\epsilon C}} + \frac{a_r^\dagger(k,\mathbf{x})\,e^{2\pi i x^- k/L}}{\sqrt{k+\epsilon C}}\right) \tag{6}$$

where

$$C = \frac{L}{2\pi\epsilon}\left(A_-(\mathbf{x}) - A_-(\mathbf{x}+\hat{r})\right) \ . \tag{7}$$

The $k=0$ term in (6) that has an imaginary square root is dropped. Consider a two-link state

$$|\Psi\rangle = \frac{1}{N}\sum_{j=0}^{K}\psi_j\,\mathrm{Tr}\left\{a_1^\dagger(j)\,a_{-1}^\dagger(K-j)\right\}\,|0\rangle\ ; \tag{8}$$

if we use (4) for the vacuum $|0\rangle$, then we have to split $|\Psi\rangle$ into a superpostion of two vacuum states

$$
\begin{aligned}
|\Psi\rangle \;&=\; \frac{1}{\sqrt{2}N}\left[\sum_{j=0}^{K-1}\psi_j^0\,\mathrm{Tr}\left\{a_1^\dagger(j,\mathbf{x})\,a_{-1}^\dagger(K-j,\mathbf{x}+\hat{1})\right\}\,|\mathbf{x},+\rangle\,|\mathbf{x}+\hat{r},-\rangle\right. \\
&\quad\left. +\sum_{j=1}^{K}\psi_j^K\,\mathrm{Tr}\left\{a_1^\dagger(j,\mathbf{x})\,a_{-1}^\dagger(K-j,\mathbf{x}+\hat{1})\right\}\,|\mathbf{x},+\rangle\,|\mathbf{x}+\hat{r},-\rangle\right]\;. \qquad (9)
\end{aligned}
$$

For one of the vacuum states, the $a_1^\dagger(0)$ mode becomes dynamical while, for the other vacuum state, the $a_{-1}^\dagger(0)$ mode becomes dynamical.

Later on, we will make an important assumption, If ϵ is small enough, we can neglect its effects for ordinary (nonzero) modes. Thus, we make the approximation that $\psi_j^0 = \psi_j^K \equiv \psi_j$ for $j \in \{1,2,\ldots,K-1\}$. The normalization of this state is

$$
\langle\Psi|\Psi\rangle = \frac{1}{2}\left|\psi_0^0\right|^2 + \frac{1}{2}\left|\psi_K^K\right|^2 + \sum_{j=1}^{K-1}|\psi_j|^2\;. \qquad (10)
$$

We see that the zero modes are "half as big" as the ordinary modes.

3 Bound state equation

Now, we can write down the bound state equation for two links. We will start by writing down[1] separate equations for ψ_j^0 and ψ_j^K:

$$
\begin{aligned}
\mathcal{M}^2\,\psi_j^0 \;&=\; \frac{\mu_B^2+\frac{g^2N}{\pi}}{(j+\epsilon)(K-j-\epsilon)}\,\psi_j^0 \\
&\quad -\frac{g^2N}{4\pi}\sum_{l=0}^{K-1}\frac{1}{(j-l)^2}\left[\frac{(j+l+2\epsilon)(2K-j-l-2\epsilon)\psi_l^0}{\sqrt{(j+\epsilon)(K-j-\epsilon)(l+\epsilon)(K-l-\epsilon)}}-4\psi_j^0\right] \qquad (11)\\
\mathcal{M}^2\,\psi_j^K \;&=\; \frac{\mu_B^2+\frac{g^2N}{\pi}}{(j-\epsilon)(K-j+\epsilon)}\,\psi_j^K \\
&\quad -\frac{g^2N}{4\pi}\sum_{l=1}^{K}\frac{1}{(j-l)^2}\left[\frac{(j+l-2\epsilon)(2K-j-l+2\epsilon)\psi_l^K}{\sqrt{(j-\epsilon)(K-j+\epsilon)(l-\epsilon)(K-l+\epsilon)}}-4\psi_j^K\right] \qquad (12)
\end{aligned}
$$

Now, we add (11) and (12) together, assuming $\epsilon \ll 1$, and write separate equations for the zero modes and the non-zero modes:

$$
\begin{aligned}
\mathcal{M}^2\,\psi_0^0 \;&=\; \frac{\mu_B^2+\frac{g^2N}{\pi}}{\epsilon K}\,\psi_0^0 - \frac{g^2N}{4\pi}\sum_{l=1}^{K-1}\frac{1}{l^2}\left[\frac{l(2K-l)\psi_l}{\sqrt{\epsilon K l(K-l)}}-4\psi_0^0\right] \qquad (13)\\
\mathcal{M}^2\,\psi_j \;&=\; \frac{\mu_B^2+\frac{g^2N}{\pi}}{j(K-j)}\,\psi_j - \frac{g^2N}{4\pi}\sum_{l=1}^{K-1}\frac{1}{(j-l)^2}\left[\frac{(j+l)(2K-j-l)\psi_l}{\sqrt{j(K-j)l(K-l)}}-4\psi_j\right]
\end{aligned}
$$

[1] I don't know how to get the $\epsilon \to 0$ limit if I use your favorite subtraction scheme.

$$
\begin{aligned}
&-\frac{g^2 N}{8\pi j^2}\left[\frac{j(2K-j)\psi_0^0}{\sqrt{\epsilon K j(K-j)}} - 4\psi_j\right] \\
&-\frac{g^2 N}{8\pi (K-j)^2}\left[\frac{j(2K-j)\psi_K^K}{\sqrt{\epsilon K j(K-j)}} - 4\psi_j\right]
\end{aligned} \tag{14}
$$

with a similar equation for ψ_K^K.

4 Solutions to the boundstate equation

One can solve (13) and (14) as a matrix eigenvalue problem. However, the associated Hamiltonian matrix is not quite Hermitian. If we rescale the zero-mode fields in the eigenvector $\mathbf{y} = \{\psi_0^0/\sqrt{2}, \psi_1, \ldots, \psi_{K-1}, \psi_K^K/\sqrt{2}\}$, the resulting eigenvalue problem

$$
\mathbf{H} \cdot \mathbf{y} = \mathcal{M}^2\, \mathbf{y} \tag{15}
$$

does have a Hermitian Hamiltonian matrix.

There are two possible strategies for numerical solution of the eigenvalue problem. The first is to keep ϵ as a small—but nonzero—parameter. If ϵ is chosen to be too small, I suspect that some sort of numerical instability occurs. If it is too large, then it is not really acting as a zero mode.

The other strategy is to rescale the zero modes

$$
\tilde{\psi}_0 = \frac{\psi_0^0}{\sqrt{\epsilon}} \quad \text{and} \quad \tilde{\psi}_K = \frac{\psi_K^K}{\sqrt{\epsilon}} \tag{16}
$$

and take the $\epsilon \to 0$ limit analytically. We obtain, in this limit:

$$
0 = \frac{\mu_B^2 + \frac{g^2 N}{\pi}}{K}\,\tilde{\psi}_0 - \frac{g^2 N}{4\pi}\sum_{l=1}^{K-1}\frac{(2K-l)\psi_l}{l\sqrt{Kl(K-l)}} \tag{17}
$$

$$
\begin{aligned}
\mathcal{M}^2\,\psi_j = {}& \frac{\mu_B^2 + \frac{g^2 N}{\pi}}{j(K-j)}\,\psi_j - \frac{g^2 N}{4\pi}\sum_{l=1}^{K-1}\frac{1}{(j-l)^2}\left[\frac{(j+l)(2K-j-l)\psi_l}{\sqrt{j(K-j)l(K-l)}} - 4\psi_j\right] \\
&-\frac{g^2 N}{8\pi j^2}\left[\frac{j(2K-j)\tilde{\psi}_0}{\sqrt{Kj(K-j)}} - 4\psi_j\right] \\
&-\frac{g^2 N}{8\pi (K-j)^2}\left[\frac{j(2K-j)\tilde{\psi}_K}{\sqrt{Kj(K-j)}} - 4\psi_j\right]
\end{aligned} \tag{18}
$$

with a similar equation for $\tilde{\psi}_K$. Note that the left hand side of (17) has vanished: this is simply the fact that the zero mode is no longer dynamical in the $\epsilon \to 0$ limit. Numerically, we have a mixture of an eigenvalue problem and a set of linear equations. One could easily write a general routine in *Mathematica* that solves such systems. I suspect that such systems can also be handled using Lanczos methods, but I haven't looked into this yet.

5 General case

In general, I have states with many link fields, some of which are zero modes. How do I modify a DLCQ calculation to include these?

1. In the Hamiltonian operator, each instance of zero momentum is replaced by ϵ.

2. Interactions between states containing p zero-modes and states containing q zero-modes are multiplied by 2^{p-q}, only when $q > p$.

3. To obtain a Hermitian matrix, states containing p zero-modes are rescaled by a factor of $2^{p/2}$.

6 Comments

1. For DLCQ, this method does not address the slow convergence problem due to the x^{β} behavior of the wavefunctions.

2. Partial improvement schemes, like the one in `hep-ph/9605409`, will not provide a consistent $\epsilon \to 0$ limit.

3. The case of two zero modes next to each other in the color string is interesting. I don't know if this formalism gets it correct, since the wavefunction is supposed to go to a constant. As a starting point, I would exclude all states containing two adjacent zero modes from the basis: this is something we can work on later.